PHY 242. Solutions to Problem Set 4

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1. Ordering with Cubic anisotropy. Consider the free energy functional

$$F = F_\circ + \frac{1}{2}t\vec{m}^2 + \frac{1}{4}u(\vec{m}^2)^2 + \frac{1}{4}v\sum_j m_j^4,$$

where $\vec{m} = (m_1, m_2, m_3)$ is the 3D order parameter (think of magnetism). The last term results in cubic anisotropy. Assume as usual that t changes sign at $T_c > 0$.

(a) Determine the phase diagram, i.e. for values of the parameters u, v that are physically allowable, find what direction the magnetization $\vec{m}$ points in. Consider only the high symmetry directions (1,0,0), (1,1,0), (1,1,1) for the direction of magnetization. Hint: in solving this problem, consider each direction separately from the beginning. Thus for the (1,1,1) direction, $m_1^2 = m_2^2 = m_3^2 = m^2/3$.

(b) What are the restrictions on u and v for a finite magnetization?

Once one has worked through the solution the following shorthand method can be figured out, but don't expect your solution to be as direct as this one. Let N=1 denote the (1,0,0) direction, N=2 denote the (1,1,0) direction, N=3 denote the (1,1,1) direction. This works because for the nonzero components of the order parameter, $m_j^2 = m^2/N$. Note, $\sum_1^3 m_j^2 = N(m^2/N) = m^2$, and $\sum_1^3 m_j^4 = N(m^2/N)^2 = m^4/N$. Then for each of the directions, the *form* of the free energy is the same, but $v \rightarrow v/N$:

$$F = F_\circ + \frac{1}{2}tm^2 + \frac{1}{4}um^4 + \frac{1}{4}vm^4/N,$$

Looking at the free energy in this form tells exactly the conditions for stability [question (b)]: $u + v/N > 0$, or $v > -Nu$. But N is not a variable; we must see which of these three conditions is the most stringent.

Case (i): $u > 0$. The most stringent is $N = 1$ (if it's true for $N = 1$ it's true for all). So $v > -u$ is the condition for a stable free energy expression. Minimizing (below T_c), $|t| = um^2 + v(m^2/N)/4$ gives $m^2 = |t|/(u + v/N)$. Plugging into the free energy,

$$\Delta F = -\frac{1}{4}\frac{t^2}{u + v/N}.$$

The boundary between (1,0,0) and (1,1,0) [N=1 and N=2] is $\frac{1}{u+v} = \frac{1}{u+v/2} \to v$=0. The boundary bewteen (1,1,0) and (1,1,1) [N=2 and N=3] is $\frac{1}{u+v} = \frac{1}{u+v/2} \to v$=0 again. All energies are equal at v=0 because for this value the free energy has spherical symmetry, i.e. the direction of $\vec{m}$ is irrelevant. Checking, for $v > 0$, N=1 (1,0,0) is the low energy solution; for $v < 0$, N=3 (1,1,1) is the solution.

Case (ii): $u < 0$. The condition is $v > N|u|$, and the most stringent is N=3. So for $u < 0$ the line $u + v/3$ is the stability limit of the system (the free energy expression). For $u \equiv 0$ N=3 is the favored "phase", i.e. (the (1,1,1) direction. For the $u < 0$ region of the phase diagram, N=3 still gives the lowest free energy.

2. Coupled Order Parameters. Consider two coupled order parameters A and B (both **real scalars**) in a Landau free energy, coupled in the simplest way and allowing no inhomogeneity (no gradients of the order parameters):

$$F = F_o + \alpha_1 A^2 + \frac{1}{2}\beta_1 A^4 + \alpha_2 B^2 + \frac{1}{2}\beta_2 B^4 + \gamma A^2 B^2.$$

Consider that, in the absence of coupling ($\gamma = 0$) there is an instability to A-type order at T_A and to B-type order at T_B:

$$\alpha_1 = a_1(T - T_A); \quad \alpha_2 = a_2(T - T_B).$$

Suppose T_A is the higher temperature, so A-type order is encountered first upon lowering the temperature.

(a) Minimize the free energy with respect to A and B, and show that there is a 2nd order transition with $A^2 = a_1^*(T_A^* - T)$. Find expressions for both the renormalized critical temperature T_A^* and the renormalized amplitude A_1^*.

Solution. First observe that A and B, if non-zero, can just as well be of one sign or the other, independently. Thus we can suppose them both to be positive; other degenerate broken symmetry states can be obtained by substituting negative values. From what is given, $\alpha_1 = a_1(T - T_A)$ defines a_1 and T_A, and as A is our primary order parameter, β_1 must be positive. Actually, β_2 must also be positive, or we cannot stop our B-expansion at 4th order. The equations are

$$\begin{aligned} \alpha_1 + \beta_1 A^2 + \gamma B^2 &= 0, \quad (1) \\ \alpha_2 + \beta_2 B^2 + \gamma A^2 &= 0, \quad (2) \end{aligned}$$

where we have assumed that A, B are non-zero. The possible types of solutions are (1) A=0, B=0 (uninteresting), (2) A and B both nonzero, this is the intended one to pursue, (3) A nonzero, B=0. A=0, B nonzero should not be relevant as we chose $T_A > T_B$. Solution (3) might be possible in some region(s) of the phase diagram, but with five constants around, this is too much of a job to consider.

So we stick to the case of both A and B nonzero, and only address the specific questions that were asked (not the phase diagram). Solving for B^2 from the 2nd equation

$$B^2 = -\frac{\gamma}{\beta_2}A^2 - \frac{\alpha_2}{\beta_2} \tag{3}$$

and substituting into the other equation lets one write in an instructive form

$$\begin{aligned}
A^2 &= \frac{\alpha_1\beta_2 - \gamma\alpha_2}{\gamma^2 - \beta_1\beta_2} && (4)\\
&= \frac{1}{\gamma^2 - \beta_1\beta_2}\Big[\beta_2 a_1(T - T_A) - \gamma a_2(T - T_B)\Big] && (5)\\
&= \frac{[\gamma a_2 T_B - \beta_2 a_1 T_A] - [\gamma a_2 - \beta_2 a_1]T}{\gamma^2 - \beta_1\beta_2} && (6)\\
&= \frac{\gamma a_2 - \beta_2 a_1}{\gamma^2 - \beta_1\beta_2}\Big(\frac{\gamma a_2 T_B - \beta_2 a_1 T_A}{\gamma a_2 - \beta_2 a_1} - T\Big) && (7)\\
&= \Big(\frac{a_1}{\beta_1}\Big)^*(T_A^* - T) && (8)
\end{aligned}$$

where we have used $\alpha_1 \equiv a_1(T - T_A), \alpha_2 \equiv a_2(T - T_B)$. The definitions of the renormalized constants $(a_1/\beta_1)^*$ and T_A^* are evident in this result. [Note a misprint in the statement of the problem.] For the realistic case, where the coupling γ is small, one has

$$\Big(\frac{a_1}{\beta_1}\Big)^* = \Big(\frac{a_1}{\beta_1}\Big)\frac{1 - \frac{\gamma a_2}{\beta_2 a_1}}{1 - \frac{\gamma^2}{\beta_2\beta_2}}, \quad T_A^* = \frac{\beta_2 a_1 T_A - \gamma a_2 T_B}{\beta_2 a_1 - \gamma a_2} = T_A\frac{1 - \frac{\gamma a_2}{\beta_2 a_1}\frac{T_B}{T_A}}{1 - \frac{\gamma a_2}{\beta_2 a_1}}. \tag{9}$$

These expressions reduces to the known results for a single order parameter when $\gamma \to 0$ (terms with B are not coupled). Thus the first effect is that the transition temperature is renormalized to T_A^*, *as long as the constants make sense, i.e.* $(a_1/\beta_1)^*$ and T_A^* must be positive.

(b) Discuss the effect of the sign of the coupling γ and its magnitude.

The correction to the amplitude of A^2 and to the critical temperature T_A are both linear in γ for small γ. For the amplitude, the numerator gives the leading correction, and a negative coupling γ actually enhances the order parameter. For the critical temperature, expansion of the denominator give the renormalization of the critical temperature as

$$T_A^* - T_A = \frac{\gamma a_2}{\beta_2 a_1}(1 - \frac{T_B}{T_A}),$$

which is an *enhancement* if γ is positive, but decreases the critical temperature if γ is negative.

(c) What are the consequences if $T_A = T_B$?

In this case the critical temperature is not changed at all (which seems pretty obvious). The critical temperatures do not enter into the expression for the amplitude.

Additional observations:
Positivity of $(a_1/\beta_1)^*$ requires $\gamma^2 < \beta_1\beta_2$ (denominator) and $\gamma^2 < \beta_2 a_1/a_2$ (numerator). This latter condition also come into the expression for T_A^*. For small $|\gamma|$ the correction is linear in γ.

Finally, it should be noted that exploration of the phase diagram has not been carried out. It might be that the solution of nonzero A but zero B might be the favored solution for some values of $a_1, a_2, \beta_1, \beta_2, \gamma$.