Solution. PHY 242 HW #2. W. Pickett

Nesting Function $\xi(Q)$ in 2D and 3D. Is a function only of $|\underset{\sim}{Q}|$ due to circular/spherical symmetry.

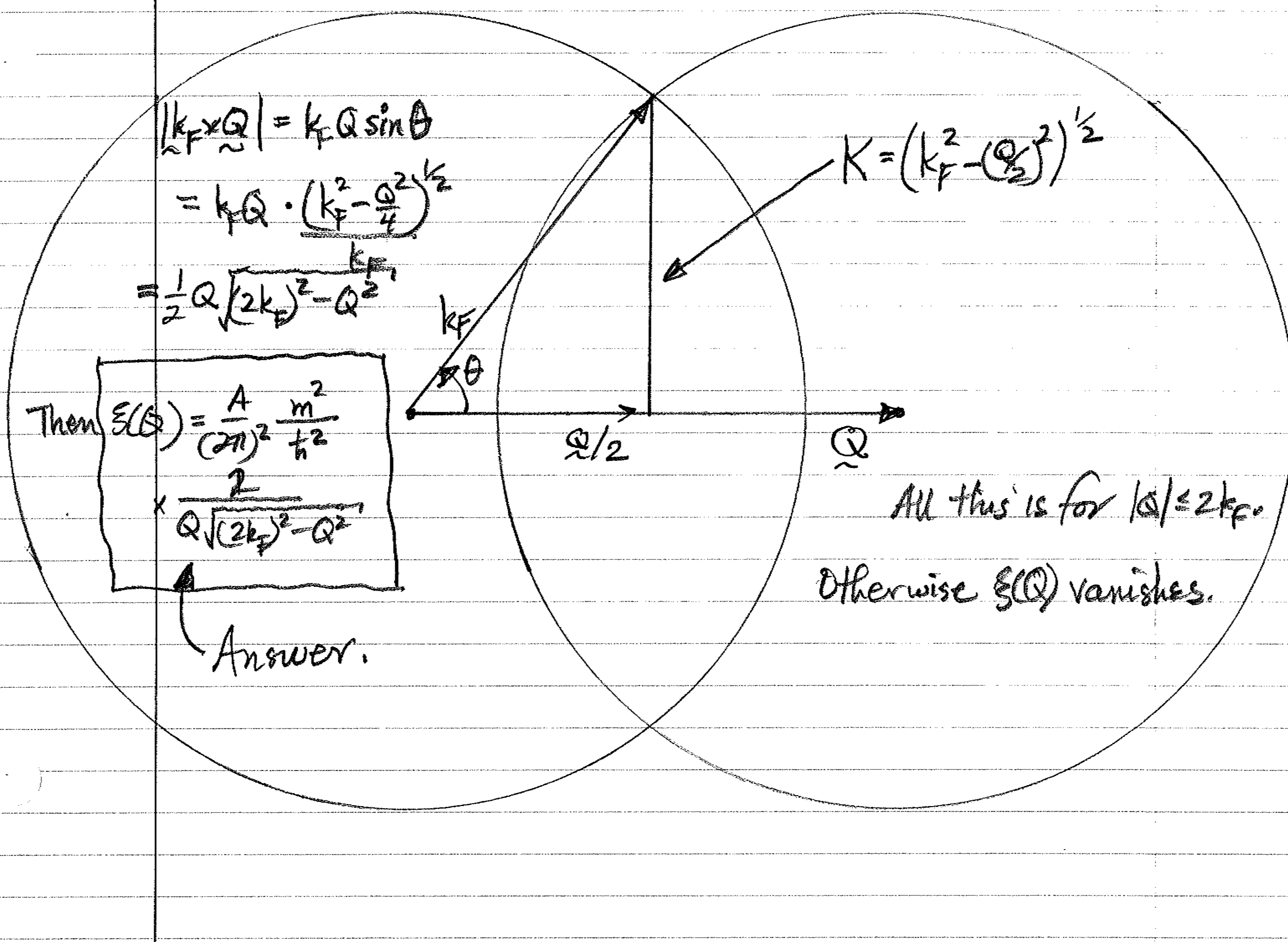
$$\xi(Q) \equiv \frac{1}{N}\sum_{k} \delta(\varepsilon_k)\,\delta(\varepsilon_{k+Q}) \quad \text{lets do } \boxed{\text{2D first}}.$$

$$= \frac{A}{(2\pi)^2}\int d^2k\, \delta(\varepsilon_k)\,\delta(\varepsilon_{k+Q}) = \frac{A}{(2\pi)^2}\int_P \frac{dP_k}{|\nabla\varepsilon_k \times \nabla\varepsilon_{k+Q}|} \quad \underline{\text{given}}$$

$\underset{\sim}{v}_k = \frac{\nabla_k \varepsilon_k}{\hbar}$

$$\rightarrow \frac{A}{(2\pi)^2}\sum_{\substack{2 \\ \text{points} \\ \text{of} \\ \text{intersection}}} \frac{1}{\hbar^2 |\underset{\sim}{v}_k \times \underset{\sim}{v}_{k+Q}|} \quad \text{which is the same at both points (see figure).}$$

So: $\varepsilon_k = \frac{\hbar^2 k^2}{2m}$, $\hbar \underset{\sim}{v}_k = \frac{\hbar \underset{\sim}{k}}{m}$ ← vector $\underset{\sim}{k}$. Cross product: $\underset{\sim}{k} \times (\underset{\sim}{k}+\underset{\sim}{Q}) = \cancel{\underset{\sim}{k}\times\underset{\sim}{k}} + \underset{\sim}{k}\times\underset{\sim}{Q}$ $= \underset{\sim}{k}\times\underset{\sim}{Q}$ on FS at point of intersection



$$|\underset{\sim}{k}_F \times \underset{\sim}{Q}| = k_F Q \sin\theta$$
$$= k_F Q \cdot \frac{\left(k_F^2 - \frac{Q^2}{4}\right)^{1/2}}{k_F}$$
$$= \frac{1}{2} Q \sqrt{(2k_F)^2 - Q^2}$$

Then
$$\boxed{\xi(Q) = \frac{A}{(2\pi)^2}\frac{m^2}{\hbar^2} \times \frac{2}{Q\sqrt{(2k_F)^2 - Q^2}}}$$
← Answer.

All this is for $|Q| \le 2k_F$.

Otherwise $\xi(Q)$ vanishes.

The 3D case: the figure (in cross section) looks exactly the same.

However, there is the same contribution all along the line of intersection, has length

$$2\pi K = 2\pi\sqrt{k_F^2 - Q^2/4} = \pi\sqrt{(2k_F)^2 - Q^2}$$

$$\xi(Q) = \frac{V}{(2\pi)^3}\int_{\mathcal{L}} \frac{d\mathcal{L}_k}{|\nabla\varepsilon_k \times \nabla\varepsilon_{k+Q}|}$$

$$= \frac{V}{(2\pi)^3}\,\frac{m^2}{\hbar^4}\,\frac{1}{Q\sqrt{(2k_F)^2-Q^2}} \times \pi\sqrt{(2k_F)^2-Q^2}$$

$$\boxed{\xi(Q) = \frac{V}{(2\pi)^3}\,\frac{\pi m^2}{\hbar^4 Q}, \text{ if } Q<2k_F\text{; zero otherwise.}}$$

What is the integral of $\xi(Q)$?

3D $$\frac{1}{N}\sum_Q \xi(Q) = \frac{V}{(2\pi)^3}\int d^3Q\, \xi(Q) = \frac{V}{(2\pi)^3}\,4\pi\int_0^{2k_F} Q^2\, dQ\, \frac{V}{(2\pi)^3}\,\frac{\pi m^2}{\hbar^4 Q}$$

$$= \left[\frac{V}{(2\pi)^3}\right]^2 4\pi^2\,\frac{m^2}{\hbar^4}\,\frac{1}{2}(2k_F)^2 = \left[\frac{V}{(2\pi)^3}\right]^2 8\pi^2\,\frac{m^2}{\hbar^4}\,k_F^2$$

$$= \left(\frac{V}{(2\pi)^3}\right)^2 \frac{2\pi^2 m^2}{\hbar^4}(2k_F)^2.$$

$$N(0) = \frac{1}{N}\sum_k \delta(\varepsilon_k) = \frac{V}{(2\pi)^3}\int d^3k\,\delta(\varepsilon_k) = \frac{V}{(2\pi)^3}\int_S \frac{dS_k}{\hbar|v_k|}$$

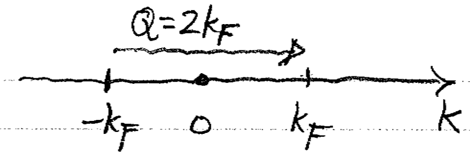
$$= \frac{V}{(2\pi)^3}\,\frac{m}{\hbar^2 k_F}\cdot 4\pi k_F^2 = \frac{V}{(2\pi)^3}\,\frac{m}{\hbar^2}\,4\pi k_F$$

$$[N(0)]^2 = \left[\frac{V}{(2\pi)^3}\right]\frac{m^2}{\hbar^4}(4\pi)^2 k_F^2 = \frac{V}{(2\pi)^3}\cdot\frac{m^2\pi^2}{\hbar^4}(2k_F)^2$$

Hmmmm... a factor of 2 error.

It must be that $\frac{1}{N}\sum_Q \xi(Q) = [N(0)]^2$.

1D



A. $\xi(Q)$ is zero except for $Q = 2k_F$, or $-2k_F$

B. $\underset{\sim}{v}_k \times \underset{\sim}{v}_{k+Q} \equiv 0$ (they are collinear) so the denominator vanishes.

C. $$\frac{1}{N}\sum_{Q} \xi(Q) = \frac{1}{N}\sum_{Q}\frac{1}{N}\sum_{k} \delta(\varepsilon_k)\,\delta(\varepsilon_{k+Q})$$
$$= \frac{1}{N}\sum_{k'}\frac{1}{N}\sum_{k} \delta(\varepsilon_k)\,\delta(\varepsilon_{k'}) = \left(N(0)\right)^2$$

D. So, $\xi(Q) \propto N(0)\,\delta(|Q| - 2k_F)$

contributions at both $Q = \pm 2k_F$ means

$$\boxed{\xi(Q) = \frac{1}{2}N(0)\,\delta(|Q| - 2k_F).}$$