Additional BCS notes.

$$b_k = c_{-k\downarrow}c_{k\uparrow} \qquad b_k^\dagger = c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \qquad \{c_{k\sigma}, c_{k'\sigma'}\} = 0$$

$$\boxed{\begin{aligned}&\{a_\alpha^\dagger, a_{\alpha'}\}_+ \quad \underline{\text{fermions}}\\ &= a_\alpha^\dagger a_{\alpha'} + a_{\alpha'}a_\alpha^\dagger\\ &= \delta_{\alpha\alpha'}(n_\alpha + (1-n_\alpha)) = +\delta_{\alpha\alpha'}\end{aligned}}$$

$$[b_k, b_{k'}^\dagger] = [c_{-k\downarrow}c_{k\uparrow}, c_{k'\uparrow}^\dagger c_{-k\downarrow}^\dagger]$$

$$= c_{-k\downarrow}c_{k\uparrow}c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger - c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger c_{-k\downarrow}c_{k\uparrow}$$

$$= c_{-k\downarrow}(\delta_{kk'} - c_{k'\uparrow}^\dagger c_{k\uparrow})c_{-k'\downarrow}^\dagger - (\qquad)$$

$$= c_{-k\downarrow}c_{-k\downarrow}^\dagger\delta_{kk'} - c_{-k\downarrow}c_{k'\uparrow}^\dagger c_{k\uparrow}c_{-k'\downarrow}^\dagger - c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger c_{-k\downarrow}c_{k\uparrow}$$

$$= (1-n_{-k\downarrow})\delta_{kk'} - c_{k'\uparrow}^\dagger c_{-k\downarrow}c_{-k'\downarrow}^\dagger c_{k\uparrow} - c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger c_{-k\downarrow}c_{k\uparrow}$$

$$= (1-n_{-k\downarrow})\delta_{kk'} - c_{k'\uparrow}^\dagger(\delta_{kk'} - c_{-k'\downarrow}^\dagger c_{-k\downarrow})c_{k\uparrow} - c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger c_{-k\downarrow}c_{k\uparrow}$$

$$= (1-n_{-k\downarrow})\delta_{kk'} - n_{k\uparrow}\delta_{kk'} + \cancel{c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger c_{-k\downarrow}c_{k\uparrow}} - \cancel{c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger c_{-k\downarrow}c_{k\uparrow}}$$

$$= (1 - n_{k\uparrow} - n_{-k\uparrow})\delta_{kk'}$$

$b_k^{\dagger 2} = 0 = b_k^2$ as for fermions. Only one pair/k ($k \equiv \{\underset{\sim}{k}, \sigma\}$)

$b_k = c_{-k\downarrow}c_{k\uparrow}$, $b_{-k} = c_{k\downarrow}c_{-k\uparrow}$ is "conjugate" pair. note

b_{-k} = spin flip of $(-c_{-k\downarrow}c_{k\uparrow}) = -b_k$

In H_{red}:

$$\sum_{ks}\varepsilon_k n_{ks} = \sum_k \varepsilon_{k\uparrow}n_{k\uparrow} + \sum_{+k}\varepsilon_{-k\downarrow}n_{-k\downarrow} = \sum_k \varepsilon_k(n_{k\uparrow}+n_{-k\downarrow}) \overset{\text{pairing approx.}}{\approx} \sum_k 2\varepsilon_k b_k^\dagger b_k$$

so

$$H^0_{red} = \sum_k 2\varepsilon_k b_k^\dagger b_k + \sum_{kk'} V_{k'k} b_{k'}^\dagger b_k$$

$$= \sum_{kk'} (2\varepsilon_k\delta_{kk'} + V_{k'k}) b_{k'}^\dagger b_k$$

$$H^0_{red} = \sum_{kk'} H^0_{red,k'k}\, b_{k'}^\dagger b_k = \sum_{k'k} b_{k'}^\dagger (H^0_{red,k'k}) b_k$$

a "single pair" operator ⇒ "solvable" (but b_k's are strange) so <u>not</u>!

$$\langle \psi_0 | \psi_0 \rangle \propto \prod_{k'} \prod_{k} \langle 0 | (1 + g_{k'}^* b_{k'})(1 + g_k b_k^\dagger) | 0 \rangle$$

no pairs in g.s.

$$= \prod_{kk'} \langle 0 | \left(1 + \cancel{g_{k'}^* b_{k'}} + \cancel{g_k b_k^\dagger} + g_{k'}^* g_k \, b_{k'} b_k^\dagger \right) | 0 \rangle$$

$$= \prod_{kk'} \left[1 + |g_k|^2 \delta_{kk'} \right] \qquad \prod_k \left(\prod_{k'} 1 \right) = \prod_k 1 = 1$$

$$= \prod_k \left(1 + |g_k|^2 \right)$$

$$= (1 + |g_{k_1}|^2)(1 + |g_{k_2}|^2)(1 + |g_{k_3}|^2) + \cdots$$ cannot simplify further

so normalized $$|\psi_0\rangle = \prod_k \frac{1 + g_k b_k^\dagger}{\sqrt{1 + |g_k|^2}} |0\rangle$$

Constrain $\langle \psi_0 | \hat{N}_{op} | \psi_0 \rangle = \langle \psi_0 | \sum_{ks} c_{ks}^\dagger c_{ks} | \psi_0 \rangle \equiv N_0$

hence minimize "functional of $\{g_k\}$"

$$\delta \langle \psi_0 | \, W \equiv \left\{ \sum_k 2(\varepsilon_k - \mu) b_k^\dagger b_k + \sum_{kk'} V_{k'k} b_{k'}^\dagger b_k \right\} | \psi_0 \rangle \equiv 0.$$

g_k are the variational parameters.

Simplify notation: $u_k \equiv \dfrac{1}{(1 + g_k^2)^{1/2}}$, $v_k = \dfrac{g_k}{(1 + g_k^2)^{1/2}}$

note that $u_k^2 + v_k^2 = 1$

and $$|\psi_0\rangle = \prod_k \left(u_k + v_k b_k^\dagger \right) |0\rangle$$

Notation (common): $\xi_k \equiv \varepsilon_k - \mu$

= energy relative to chemical potential.

$$W = \sum_k 2\xi_k v_k^2 + \sum_{kk'} V_{k'k} u_k v_k u_{k'} v_{k'} = W_1 + W_2$$

Let $u_k = \sin\beta_k$, $v_k = \cos\beta_k$ $\Big\{ \sin x \cos x = \frac{1}{2}\sin 2x,$
$\cos^2 x - \sin^2 x = \cos 2x$
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$W_1 = \sum_{k'} 2\xi_{k'} \cos^2\beta_{k'}$$

$$\frac{\delta W_1}{\delta\beta_k} = -4\xi_k \sin\beta_k \cos\beta_k = -2\xi_k \sin 2\beta_k$$

$$\frac{\delta W_2}{\delta\beta_k} = \frac{\delta}{\delta\beta_k}\sum_{k''k'} V_{k''k'} u_{k'} v_{k'} u_{k''} v_{k''},$$ 2 identical contributions from k' and k'' quantities

$$= 2\sum_{k'} V_{k'k} u_{k'} v_{k'} \frac{d}{d\beta_k}\left(\frac{1}{2}\sin 2\beta_k\right)$$

$$= 2\sum_{k'} V_{k'k} u_{k'} v_{k'} \cos 2\beta_k$$

$$\frac{\delta W}{\delta\beta_k} = 0 \Rightarrow -\frac{\delta W_1}{\delta\beta_k} = \frac{\delta W_2}{\delta\beta_k}$$

$\frac{1}{2}\sin 2\beta_{k'}$

$$2\xi_k \sin 2\beta_k = \left(\sum_{k'} V_{k'k} u_{k'} v_{k'}\right)\cos 2\beta_k$$

or

$$\cot 2\beta_k = \frac{4\xi_k}{\sum_{k'} V_{k'k} \sin 2\beta_{k'}}$$ self-consistency condition

Sol'n:

$$u_k^2 = \frac{1}{2}\left(1+\frac{\xi_k}{E_k}\right), \quad v_k^2 = \frac{1}{2}\left(1-\frac{\xi_k}{E_k}\right), \quad u_k v_k = \frac{\Delta_k}{2E_k}$$

with $E_k = +(\xi_k^2 + \Delta_k^2)^{1/2}$ where

$$\Delta_k = -\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{2E_{k'}}$$ <u>BCS Gap Equation</u>

or

$$\sum_{k'}\left\{\delta_{k',k} + \sum_{k'} V_{kk'}\frac{1}{2E_{k'}}\right\}\Delta_{k'} = 0$$

Find zero eigenvalues of this matrix.