

6 Phys 240C Homework 6, Electronic Response

For a 3d gas of non-interacting electrons, the energy is

$$\begin{aligned}\epsilon(\vec{k}) &= \frac{\hbar^2 k^2}{2m}, \text{ so} \\ \epsilon(\vec{k} + \vec{Q}) - \epsilon(\vec{k}) &= \frac{\hbar^2 Q^2}{2m} + \frac{\hbar^2}{m} \vec{k} \cdot \vec{Q}\end{aligned}$$

The density-density response function is

$$\chi_0(\vec{Q}, \omega) = 2 \int \frac{d^3 k}{(2\pi)^3} \frac{f(\epsilon_k) - f(\epsilon_{k+Q})}{\epsilon(\vec{k} + \vec{Q}) - \epsilon(\vec{k}) - \hbar\omega - i\eta}$$

where the factor of 2 is from the sum over spin states. The numerator can be rewritten

$$f(\epsilon_k) - f(\epsilon_{k+Q}) = f(\epsilon_k)(1 - f(\epsilon_{k+Q})) - f(\epsilon_{k+Q})(1 - f(\epsilon_k))$$

and I can make the substitution $k' = -k - Q$ in the second term and use the fact that $\epsilon(-\vec{k}) = \epsilon(\vec{k})$ to obtain

$$\begin{aligned}\chi_0(\vec{Q}, \omega) &= 2 \int \frac{d^3 k}{(2\pi)^3} \left[\frac{f(\epsilon_k)(1 - f(\epsilon_{k+Q}))}{\epsilon(\vec{k} + \vec{Q}) - \epsilon(\vec{k}) - \hbar\omega - i\eta} - \frac{f(\epsilon_{k+Q})(1 - f(\epsilon_k))}{\epsilon(k + Q) - \epsilon(k) - \hbar\omega - i\eta} \right] \\ &= 2 \int \frac{d^3 k}{(2\pi)^3} \left[\frac{f(\epsilon_k)(1 - f(\epsilon_{k+Q}))}{\epsilon(\vec{k} + \vec{Q}) - \epsilon(\vec{k}) - \hbar\omega - i\eta} - \frac{f(\epsilon_{-k})(1 - f(\epsilon_{-k-Q}))}{\epsilon(-\vec{k}) - \epsilon(-\vec{k} - \vec{Q}) - \hbar\omega - i\eta} \right] \\ &= 2 \int \frac{d^3 k}{(2\pi)^3} \left[\frac{f(\epsilon_k)(1 - f(\epsilon_{k+Q}))}{\epsilon(\vec{k} + \vec{Q}) - \epsilon(\vec{k}) - \hbar\omega - i\eta} - \frac{f(\epsilon_k)(1 - f(\epsilon_{k+Q}))}{\epsilon(\vec{k}) - \epsilon(\vec{k} + \vec{Q}) - \hbar\omega - i\eta} \right] \\ &= 2 \int \frac{d^3 k}{(2\pi)^3} f(\epsilon_k)(1 - f(\epsilon_{k+Q})) \left[\frac{1}{\hbar\omega + \hbar\omega_{kQ} + i\eta} - \frac{1}{\hbar\omega - \hbar\omega_{kQ} + i\eta} \right]\end{aligned}$$

where I have defined $\hbar\omega_{kQ} = \epsilon(\vec{k} + \vec{Q}) - \epsilon(\vec{k}) = \hbar^2 Q^2 / 2m + \vec{k} \cdot \vec{Q} / m$. For the real part, the fraction can be combined by finding a common denominator and taking the limit as $\eta \rightarrow 0$ to get

$$\lim_{\eta \rightarrow 0} \text{Re } \chi_0(\vec{Q}, \omega) = -\frac{4}{\hbar} \int \frac{d^3 k}{(2\pi)^3} \frac{[f(\epsilon_k) - f(\epsilon_k)f(\epsilon_{k+Q})] \omega_{kQ}}{\omega^2 - \omega_{kQ}^2}$$

The second term involving the product of Fermi functions is odd under the exchange of $\vec{k} \leftrightarrow \vec{k} + \vec{Q}$ so after a little manipulation, it can be shown that this term vanishes. I need to (1) swap the arguments k and $k + Q$, (2) set $k = -k' - Q$ and then (3) use the fact that for this particular function, $f(-k, -k - Q) = f(k, k + Q)$. For example, in $d = 1$,

$$\int_{-\infty}^{\infty} f(k, k + Q) dk = - \int_{-\infty}^{\infty} f(k + Q, k) dk \quad (6.1)$$

$$= - \int_{-\infty}^{\infty} f(-k', -k' - Q) dk' \quad (6.2)$$

$$= - \int_{-\infty}^{\infty} f(k', k' + Q) dk' \quad (6.3)$$

so the integral is zero. I am now left with

$$\begin{aligned}\lim_{\eta \rightarrow 0} \text{Re } \chi_0(\vec{Q}, \omega) &= -\frac{4}{\hbar} \int \frac{d^3 k}{(2\pi)^3} \frac{f(\epsilon_k) \omega_{kQ}}{\omega^2 - \omega_{kQ}^2} \\ &= 2 \int \frac{d^3 k}{(2\pi)^3} f(\epsilon_k) \left[\frac{1}{\hbar\omega + \hbar\omega_{kQ}} - \frac{1}{\hbar\omega - \hbar\omega_{kQ}} \right] \\ &= \frac{2}{(2\pi)^2} \int_0^{k_F} k^2 dk \int_0^\pi \sin \theta d\theta \left[\frac{1}{\hbar\omega + \hbar\omega_{kQ}} - \frac{1}{\hbar\omega - \hbar\omega_{kQ}} \right]\end{aligned}$$

The remainder of this derivation can be followed in Fetter & Walecka's "Quantum Theory of Many-Particle Systems" or similarly in section V of "Lindhard function of a d -dimensional Fermi gas" by Bogdan Mihaila available here: <http://arxiv.org/pdf/1111.5337v1.pdf>

Next, I will define the dimensionless quantities $\xi = k/k_F$, $q = Q/k_F$, $\nu = \hbar\omega/2\epsilon_F$ and $x = \cos\theta$ to get

$$\begin{aligned}\hbar\omega_{kQ} &= \frac{\hbar^2}{m}(Q^2/2 + \vec{k} \cdot \vec{Q}) \\ &= \frac{\hbar^2}{m}k_F^2(q^2/2 + \xi qx) \\ &= 2\epsilon_F(q^2/2 + \xi qx) \\ \lim_{\eta \rightarrow 0} \text{Re } \chi_0(\vec{Q}, \omega) &= \frac{k_F^3}{\epsilon_F(2\pi)^2} \int_0^1 \xi^2 d\xi \int_{-1}^1 dx \left[\frac{1}{\nu + q^2/2 + \xi qx} - \frac{1}{\nu - q^2/2 - \xi qx} \right] \\ &= \frac{1}{2}D(\epsilon_F) \int_0^1 \xi^2 d\xi \int_{-1}^1 dx \left[\frac{1}{\nu + q^2/2 + \xi qx} - \frac{1}{\nu - q^2/2 - \xi qx} \right]\end{aligned}$$

where $D(\epsilon_F) = mk_F/\hbar^2\pi^2$ is the density of states at the Fermi energy. The remainder of the problem involves solving this integral using the following relations

$$\begin{aligned}\lim_{\eta \rightarrow 0} \text{Re} \int_{-1}^1 dx \frac{1}{ax - b - i\eta} &= \text{P.V.} \int_{-1}^1 dx \frac{1}{ax - b} \\ &= \frac{1}{a} \ln \left| \frac{b-a}{b+a} \right| \\ \int_0^1 x dx \ln |ax + b| &= \frac{b}{2a} - \frac{1}{4} + \frac{1}{2} \left(1 - \frac{b^2}{a^2} \right) \ln |a + b|\end{aligned}$$

I will just quote the final result:

$$\lim_{\eta \rightarrow 0} \text{Re } \chi_0(\vec{Q}, \omega) = \frac{1}{2}D(\epsilon_F) \left\{ -1 + \frac{1}{2q} [1 - (\xi^-)^2] \ln \left| \frac{1 + \xi^-}{1 - \xi^-} \right| - \frac{1}{2q} [1 - (\xi^+)^2] \ln \left| \frac{1 + \xi^+}{1 - \xi^+} \right| \right\}$$

where I have defined $\xi^\pm = v/q \pm q/2$.

Refer to the references above for the imaginary part and the $d = 1$ case.