Physics 240B: Solution to Problem Set 1
Due: January 14, 2008

1. Number Operator Algebra. 10 points.
For a fermionic number annihilation operator a and its adjoint, the creation operator $a^\dagger$, the number operator is $\hat{n} = a^\dagger a$. Demonstrate that $\hat{n}^2 = \hat{n}$. [Note: two operators are equivalent if each of their matrix elements are identical.]
Solution. For 2nd quantized operators $a, a^\dagger$ (which normally have indices), the number operator is $\hat{n} = a^\dagger a$. For fermions, $\{a, a^\dagger\} = 1$ and $a^2 = 0$, i.e. it has only zero matrix elements because when a operators twice on a state, it will annihilate it even the state corresponding to a was occupied. Then

$$\hat{n}^2 = a^\dagger a a^\dagger a = a^\dagger[1 - aa^\dagger]a^\dagger = a^\dagger a - a^\dagger a^2 a^\dagger = \hat{n}.$$

2. Second Quantization of a Two State System. 40 points.
Consider a system of fermions for which there are two single-particle states, call them a and b.
(i) **Obtain the matrix representations** of the creation/annihilation operators a, b, analogous to what was done for the one-state system in class.
(ii) find the representations of the product operators $b^\dagger b, a^\dagger a, ab^\dagger$. Provide an interpretation of the first two of these.
Recall that, for the one-state problem,

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad a^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad N = a^\dagger a = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

Note that, for two particles, there are four "configurations" (states in 2nd-quantizationland), corresponding to $n_a = 0$ or 1, $n_b = 0$ or 1. Your operators representations of a, b should do exactly, and only, what you need them to do.

Hint: to begin, specify your states specifically and symbolically, call them
$\phi_1 \leftrightarrow n_a = 0 = n_b$,
$\phi_2 \leftrightarrow n_a = 1, n_b = 0$,
$\phi_3 \leftrightarrow n_a = 0, n_b = 1$,
$\phi_4 \leftrightarrow n_a = 1 = n_b$.
Then use what you know the the matrix elements need to be to construct the representations.

Solution. Make definitions specific:

$$\begin{aligned}
\text{vacuum with no particles}: \quad & |0> && \equiv |0,0) = |\phi_1> \\
\text{one ``a}'' \text{ particle}: \quad & a^\dagger|0> && \equiv |1,0) = |\phi_2> \\
\text{one ``b}'' \text{ particle}: \quad & b^\dagger|0> && \equiv |0,1) = |\phi_3> \\
\text{one ``b}'', \underline{\text{then}} \text{ one``a}'' :: \quad & a^\dagger b^\dagger|0> && \equiv |1,1) = |\phi_4>
\end{aligned}$$

Then choose a specific (orthonormal) representation of the states.

$$|\phi_1>= \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \qquad |\phi_2>= \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \qquad |\phi_3>= \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \qquad |\phi_4>= \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \tag{2}$$

The subscript on ϕ denotes the order of the state in our list.

Now consider the action of (say) $a^\dagger$:

$$a^\dagger|\phi_1>=|\phi_2>; \quad a^\dagger|\phi_2>=0; \quad a^\dagger|\phi_3>=|\phi_4>; \quad a^\dagger|\phi_4>=0.$$

The matrix elements of the representation of $a^\dagger$ are obtained by projecting these *kets* onto each of the *bras* in the basis. $<\phi_j|a^\dagger|\phi_1>=<\phi_j|\phi_2>=\delta_{j,2}$, i.e. all of these four matrix elements are zero except for $j=2$, which is one. Doing the same projections for the other three actions of $a^\dagger$ above then leads to

$$(a^\dagger)_{i,j} = \begin{bmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix} \tag{3}$$

Now we do the same for $b^\dagger$.

$$b^\dagger|\phi_1>=|\phi_2>; \quad a^\dagger|\phi_2>=-|\phi_3>; \quad b^\dagger|\phi_3>=0; \qquad b^\dagger|\phi_4>=0.$$

Forming the matrix elements in the same way leads to

$$(b^\dagger)_{i,j} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&-1&0&0\end{bmatrix} \tag{4}$$

Note that there is nothing very systematic about the representation of $b^\dagger$ compared to $a^\dagger$, it is just necessary to work it out. Notice especially the *minus sign* in one of the nonzero elements of $b^\dagger$. This is the evidence (the only evidence) of the anticommutation of $a^\dagger$ and $b^\dagger$, i.e. the fermionic nature of the operators.

Now it is easy to multiply and obtain

$$a^\dagger a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad b^\dagger b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{5}$$

These operators are *number operators* and their matrix elements are easy to understand. They are diagonal, and the number in the diagonal reflects the number of particles (a in the first case; b in the second case) in the corresponding state.

The matrix representation of $ab^\dagger$ is obtained by taking the adjoint of $a^\dagger$ and multiplying it with $b^\dagger$, with result

$$ab^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

One should check that $ab^\dagger = -b^\dagger a$ as it should be.

Its form is a mystery until you operate with it on an arbitrary state $|\psi>$:

$$ab^\dagger|\psi> = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\psi_2 \\ 0 \end{bmatrix} = -\psi_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \psi_2|\phi_3> . \tag{7}$$

Only beginning with a state with some admixture of $|\phi_3>$ (with amplitude ψ_3 in our arbitrary state) can one add a b particle and destroy an a particle. This verifies that the matrix representation makes sense. The minus sign doesn't have special physical significance, but if it's not there then the result is wrong!