Physics 240B: Solution to Problem Set 1
Due: January 14, 2008

1. **Number Operator Algebra.** 10 points.
For a fermionic number annihilation operator $a$ and its adjoint, the creation operator $a^\dagger$, the number operator is $\hat{n} = a^\dagger a$. Demonstrate that $\hat{n}^2 = \hat{n}$. [Note: two operators are equivalent if each of their matrix elements are identical.]

**Solution.** For 2nd quantized operators $a, a^\dagger$ (which normally have indices), the number operator is $\hat{n} = a^\dagger a$. For fermions, $\{a, a^\dagger\} = 1$ and $a^2 = 0$, i.e. it has only zero matrix elements because when $a$ operators twice on a state, it will annihilate it even the state corresponding to $a$ was occupied. Then

$$\hat{n}^2 = a^\dagger aa^\dagger = a^\dagger [1 - aa^\dagger] a^\dagger = a^\dagger a - a^\dagger a^2 a^\dagger = \hat{n}.$$

2. **Second Quantization of a Two State System.** 40 points.
Consider a system of fermions for which there are two single-particle states, call them $a$ and $b$.

(i) **Obtain the matrix representations** of the creation/annihilation operators $a, b$, analogous to what was done for the one-state system in class.

(ii) find the representations of the product operators $b^\dagger b, a^\dagger a, ab^\dagger$. Provide an interpretation of the first two of these.

Recall that, for the one-state problem,

$$a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad a^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad N = a^\dagger a = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (1)$$

Note that, for two particles, there are four “configurations” (states in 2nd-quantization land), corresponding to $n_a = 0$ or $1$, $n_b = 0$ or $1$. Your operators representations of $a, b$ should do exactly, and only, what you need them to do.

**Hint:** to begin, specify your states specifically and symbolically, call them

$\phi_1 \leftrightarrow n_a = 0 = n_b,$
$\phi_2 \leftrightarrow n_a = 1, n_b = 0,$
$\phi_3 \leftrightarrow n_a = 0, n_b = 1,$
$\phi_4 \leftrightarrow n_a = 1 = n_b.$

Then use what you know the the matrix elements need to be to construct the representations.
Solution. Make definitions specific:

vacuum with no particles: $|0\rangle \equiv |0,0\rangle = |\phi_1\rangle$

one "a" particle: $a^\dagger |0\rangle \equiv |1,0\rangle = |\phi_2\rangle$

one "b" particle: $b^\dagger |0\rangle \equiv |0,1\rangle = |\phi_3\rangle$

one "b", then one "a": $a^\dagger b^\dagger |0\rangle \equiv |1,1\rangle = |\phi_4\rangle$

Then choose a specific (orthonormal) representation of the states.

$$
|\phi_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
|\phi_2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},
|\phi_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},
|\phi_4\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
$$

(2)

The subscript on $\phi$ denotes the order of the state in our list.

Now consider the action of (say) $a^\dagger$:

$$
a^\dagger |\phi_1\rangle = |\phi_2\rangle; ~ a^\dagger |\phi_2\rangle = 0; ~ a^\dagger |\phi_3\rangle = |\phi_4\rangle; ~ a^\dagger |\phi_4\rangle = 0.
$$

The matrix elements of the representation of $a^\dagger$ are obtained by projecting these kets onto each of the bras in the basis. $<\phi_j|a^\dagger |\phi_1\rangle = <\phi_j|\phi_2\rangle = \delta_{j,2}$, i.e. all of these four matrix elements are zero except for $j = 2$, which is one. Doing the same projections for the other three actions of $a^\dagger$ above then leads to

$$
(a^\dagger)_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(3)

Now we do the same for $b^\dagger$.

$$
b^\dagger |\phi_1\rangle = |\phi_2\rangle; ~ a^\dagger |\phi_2\rangle = -|\phi_3\rangle; ~ b^\dagger |\phi_3\rangle = 0; ~ b^\dagger |\phi_4\rangle = 0.
$$

Forming the matrix elements in the same way leads to

$$
(b^\dagger)_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}
$$

(4)

Note that there is nothing very systematic about the representation of $b^\dagger$ compared to $a^\dagger$, it is just necessary to work it out. Notice especially the minus sign in one of the
nonzero elements of $b^\dagger$. This is the evidence (the only evidence) of the anticommutation of $a^\dagger$ and $b^\dagger$, i.e. the fermionic nature of the operators.

Now it is easy to multiply and obtain

$$a^\dagger a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b^\dagger b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

These operators are *number operators* and their matrix elements are easy to understand. They are diagonal, and the number in the diagonal reflects the number of particles ($a$ in the first case; $b$ in the second case) in the corresponding state.

The matrix representation of $ab^\dagger$ is obtained by taking the adjoint of $a^\dagger$ and multiplying it with $b^\dagger$, with result

$$ab^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

One should check that $ab^\dagger = -b^\dagger a$ as it should be.

Its form is a mystery until you operate with it on an arbitrary state $|\psi >$:

$$ab^\dagger |\psi > = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\psi_2 \\ 0 \end{bmatrix} = -\psi_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \psi_2 |\phi_3 >. \tag{7}$$

Only beginning with a state with some admixture of $|\phi_3 >$ (with amplitude $\psi_3$ in our arbitrary state) can one add a $b$ particle and destroy an $a$ particle. This verifies that the matrix representation makes sense. The minus sign doesn’t have special physical significance, but if it’s not there then the result is wrong!