Weyl eq'n: two component wvfn. Lorentz covariance.

$\sigma^\mu \partial_\mu \psi = 0 \quad : \sum_{j=0}^{3} \sigma_j \frac{\partial}{\partial x_j} \psi = 0 \; ; \quad x_0 = ct \, , \; \sigma_0 = I^{(2)}$

$\frac{\partial}{\partial x_0} = \frac{1}{c}\frac{\partial}{\partial t}$

Explicitly
$$\left[\begin{array}{c|c} \partial_0 + \partial_z & \partial_x - i\partial_y \\ \hline \partial_x + i\partial_y & \partial_0 - \partial_z \end{array}\right] \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = 0$$

Try
$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \chi \, e^{-i(k\cdot r - \omega t)} \equiv \chi \, e^{-i(\underset{\sim}{p}\cdot\underset{\sim}{r} - Et)/\hbar}$$

Then
$$i \begin{bmatrix} \frac{\omega}{c} - k_z & k_x - ik_y \\ k_x + ik_y & \omega/c + k_z \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = 0 .$$

Eigvals λ
$$\left[\left(\frac{\omega}{c} - k_z\right) - \lambda\right]\left[\left(\frac{\omega}{c} + k_z\right) - \lambda\right] - (k_x + ik_y)(k_x - ik_y) = 0$$

$$\left(\frac{\omega^2}{c^2} - k_z^2\right) - \lambda\left[\frac{\omega}{c} + \cancel{k_z} + \frac{\omega}{c} - \cancel{k_z}\right] + \lambda^2 - \left(k_x^2 + k_y^2\right) = 0$$

$$\lambda^2 + \left(\frac{\omega^2}{c^2} - \underset{\sim}{k}^2\right) - \lambda \, 2\frac{\omega}{c} = 0$$

In case one wants to diagonalize the matrix.

$$\lambda = \frac{1}{2}\left\{ 2\frac{\omega}{c} \pm \left[4\frac{\omega^2}{c^2} + 4\left(\underset{\sim}{k}^2 - \frac{\omega^2}{c^2}\right)\right]^{1/2} \right\}$$

$$= \frac{\omega}{c} \pm \left[\frac{\omega^2}{c^2} + \underset{\sim}{k}^2 - \frac{\omega^2}{c^2}\right]^{1/2} = \frac{\omega}{c} \pm |\underset{\sim}{k}|$$

Det[ ] = 0 $\Rightarrow$ $\boxed{\omega = \pm c|\underset{\sim}{k}|}$ Linear dispersion, massless

$\omega = \pm \omega_k = c|\underset{\sim}{k}|$

Right and Left handed spinors can be defined

$\sigma^\mu \partial_\mu \psi_R = 0$

$\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \qquad \bar{\sigma}^\mu = (I^{(2)}, -\sigma_x, -\sigma_y, -\sigma_z)$

Helicity: projection of $\underset{\sim}{J} = \underset{\sim}{L} + \underset{\sim}{S}$ onto $\underset{\sim}{p}$: $\underset{\sim}{p} \cdot \underset{\sim}{J}$

Easy to show: $\underset{\sim}{p} \cdot \underset{\sim}{L}\, e^{-i\underset{\sim}{k} \cdot \underset{\sim}{r}} = \underset{\sim}{p} \cdot \underset{\sim}{r} \times \underset{\sim}{p}\, e^{-ik \cdot r} = 0$

so helicity here is $\underset{\sim}{p} \cdot \underset{\sim}{S}$; helicity is conserved.

$\underset{\sim}{p} \cdot \underset{\sim}{S} \,|\underset{\sim}{p}, \lambda\rangle = \lambda |\underset{\sim}{p}|\, |\underset{\sim}{p}, \lambda\rangle$ or $\hat{p} \cdot \underset{\sim}{S} \,|\underset{\sim}{p}, \lambda\rangle = \lambda |\underset{\sim}{p}, \lambda\rangle$, $\lambda = \pm \frac{1}{2}$.

$S_z = \pm \frac{1}{2} \Rightarrow$ spin-half massless particle.

Neutrinos <u>used</u> <u>to</u> <u>be</u> described by Weyl eq'n, until discovery that they have mass (very tiny mass).

[before Schrödinger] 1926.

Klein-Gordon eq'n: 1925, proposed to describe electrons

Note: Weyl eq'n is 1st order in derivatives: $(\partial_0 + \underset{\sim}{\sigma}\cdot\underset{\sim}{\partial})\psi = 0$

Observe: $(\partial_0 - \sigma\cdot\underset{\sim}{\partial})(\partial_0 + \sigma\cdot\underset{\sim}{\partial}) = \partial_0^2 - \underset{\sim}{\partial}^2$ gives wv eq'n,

(not $(\partial_0 + \sigma\cdot\partial)^2$, which is weird)

① K-G: $\left(\frac{\partial^2}{\partial x_0^2} - \underset{\sim}{\nabla}^2\right)\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$ mass m

Often written as $[\Box + \mu^2]\psi$, $\mu = \frac{mc}{\hbar}$ = mass in nat'l units.

d'Alembert operator $\Box \equiv -\eta^{\mu\nu}\partial_\mu\partial_\nu$, $\eta^{\mu\mu} = (-,+,+,+)$ diagonal

So $(-\partial_t^2 + \nabla^2)\psi = m^2\psi$, $[c = 1 = \hbar]$

$\psi = e^{i(k\cdot r - \omega t)} \Rightarrow E_k^2 = p^2 + m^2$, $E_k = \pm\sqrt{\underset{\sim}{k}^2 + m^2}$.

Does not describe spin. Useful for spinless mesons (for example).

DERIVATION. Fock, Kudar, deDonder, deBroglie, ...

Non-rel: $\frac{p^2}{2m} = E$. QM: $\frac{\underset{\sim}{P}^2}{2m}\psi = E\psi$; $\underset{\sim}{P} \to -i\hbar\underset{\sim}{\nabla}$, $E \to i\hbar\frac{\partial}{\partial t}$ energy operator

Spec. relativity: $\sqrt{p^2c^2 + m^2c^4} = E \rightarrow \sqrt{\underset{\sim}{P}^2c^2 + m^2c^4}\,\psi = i\hbar\frac{\partial}{\partial t}\psi$

but space- & time derivatives enter differently; E&M fields would not work. (Dirac)

K-G just squared the eq'n, getting ① above

K-G eq'n does give a conserved current.