PHY 215C, QM Solutions to Homework Set #7

Due: June 1, 2017

June 4, 2017

1. Angular Momentum in Dirac Theory. 20 points.
Given the Dirac Hamiltonian $H_D = c\alpha \cdot \vec{p} + \beta mc^2 + V(r)$ with a spherical potential:
(i) Show that $[\vec{L}, H_D]$ is non-zero, so the orbital angular momentum is not a constant of the motion.
(ii) show that $[\vec{S}, H_D]$ also is not zero, but is such that the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ does commute, and therefore is a constant of the motion. The spin operator is given by

$$\vec{S} = \frac{\hbar}{2} \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}.$$

Solution; (i) Orbital angular momentum, Use $H_D = c\vec{\alpha} \cdot \vec{p} + \beta mc^2 + V(r)$.
$[\vec{L}, H_D] = [\vec{L}, c\vec{\alpha} \cdot \vec{p}] = c[\vec{r} \times \vec{p}, \vec{\alpha} \cdot \vec{p}] = c[\vec{r}, \alpha \cdot \vec{p}] \times \vec{p}$, (since $\beta mc^2 =$ constant, $[\vec{L}, f(r)] = 0$).
Now, $[\vec{r}, \vec{p}] = i\hbar I$, thus $[\vec{L}, H_D] = i\hbar c\vec{\alpha} \times \vec{p}$.

(ii) Spin angular momentum, $[\vec{S}, H_D] = [\vec{S}, c\vec{\alpha} \cdot \vec{p}]$,
z-component: $\begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix} \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix} = 2i(\alpha_2, -\alpha_1, 0)$.
So, $[\vec{S}, H_D]_z = c\frac{1}{2}\hbar 2i[\alpha_2 p_1 - \alpha_1 p_2] = -i\hbar c(\vec{\alpha} \times \vec{p})_z$, or $[\vec{S}, H_D] = -i\hbar c(\alpha \times \vec{p})$, exactly the negative of $[\vec{L}, H_D]$.
Thus $\vec{L} + \vec{S}$ commutes with H_D and is conserved for $V(|\vec{r}|)$ and $\vec{A} = 0$.

2. Relativistic Electron in a Harmonic Potential. 25 points.
Consider a Dirac electron confined to a 3D harmonic oscillator potential $V(\vec{r}) = (m\omega^2/2)\vec{r}^2$. You should realize that the derivation of relativistic corrections in Shankar Sec. 20.2 applies to general spherical potential wells, not just the H atom, so you don't need to do that again. Using Eq. 20.2.28:
(i) evaluate the relativistic correction to the kinetic energy (p^4 term) in the ground state, expressed as a fraction of the zero point energy. Use (of course) the non-relativistic eigenstates.
(ii) work out the explicit form of the spin-orbit term for this harmonic oscillator (4th term in Eq. 20.2.28).
(iii) Argue, "semiclassically" if you want, what will be the lowest state with a non-zero expectation value of the spin-orbit interaction.

Solution; The lowest order kinetic energy correction is $-\frac{p^4}{8m^3c^2}$. For the harmonic oscillator $P_i = i(\frac{m\omega\hbar}{2})^{1/2}(a_i^+ - a_i)$. So evaluate:
$< 0,0,0|P^4|0,0,0 >=< 0,0,0|\Sigma_{i,j=1}^3(p_i^4 + 2p_ip_j)|0,0,0 >= \frac{15}{4}(m\omega\hbar)^2$.
Then $\frac{-<\frac{p^4}{8m^3c^2}>}{(3/2)\hbar\omega} = \frac{-5}{16}\frac{\hbar\omega}{mc^2}$.

(ii) The explicit form of spin-orbit coupling
$\frac{-\vec{\sigma}\cdot\vec{p}\times[\vec{p},V(r)]}{4m^2c^2} = \frac{-\vec{\sigma}\cdot\vec{p}\times[\vec{p},\frac{1m\omega^2r^2}{2}]}{4m^2c^2} = \frac{-\hbar m\omega^2}{8m^2c^2}\vec{\sigma}\cdot\vec{p}\times[\vec{\nabla},r^2] = \frac{-\hbar\omega^2}{4mc^2}\vec{\sigma}\cdot\vec{p}\times\vec{r}$
$= \frac{-\hbar\omega^2}{4mc^2}\vec{\sigma}\cdot\vec{p}\times\vec{r} = \frac{-\hbar\omega^2}{4mc^2}\vec{\sigma}\cdot\vec{L} = \frac{-\omega^2}{2mc^2}\vec{S}\cdot\vec{L} = -\hbar\omega\frac{\hbar\omega}{mc^2}\frac{\vec{S}}{\hbar}\cdot\frac{\vec{L}}{\hbar}$.
The latter product operator is "of order unity" and the constants are separated to give a quick idea of the magnitude.

(iii) "Semiclassically" one expects that a single excitation, say the x one $|1,0,0>$ describes a pendumum swinging back and forth along the x-axis, and this motion would have no angular momentum about the origin. However, this first excitation is 3fold degenerate, and one can form a more general eigenstate (still an eigenstate) using the form $|1,0,0> +e^{i\phi}|0,1,0> +e^{i\chi}|0,0,1>$ (one phase can be chosen as unity) and because the resulting "motions" are out of phase, there will be some angular momentum. This can be verified by calculation. For example, using x and y states 120 degrees out of phase will give an $m = \pm1$ state (I am guessing, I haven't worked it out, but 360/3=120). Another viewpoint: adding in these phases produces an orbital current which in general has some circulation around the origin, and hence some angular momentum. Of course, calculating $<\psi|\vec{L}|\psi>$ is easy.

3. Electromagnetics in Dirac Theory. 15 points.
Prove two operator identities that are used in Sec. 20.2 (and elsewhere in physics).
(a) For vector operators $\vec{C}, \vec{D}$ that commute with the Pauli matrices $\vec{\sigma}$ (such as $\vec{r},\ \vec{p},\ \vec{L}$, etc.), show that

$$\sigma \cdot \vec{C} \sigma \cdot \vec{D} = \vec{C} \cdot \vec{D} + i\sigma \cdot \vec{C} \times \vec{D}.$$

(b) Show that

$$\Pi \times \Pi = \frac{iq\hbar}{c}\vec{B}, \qquad \text{where} \qquad \Pi \equiv \vec{p} - \frac{q\vec{A}}{c}.$$

Here $\vec{B}, \vec{A}$ are the magnetic field and vector potential, respectively.
Solution. (a)

$$\begin{aligned}
\sigma_i \cdot \vec{C}\, \sigma_j \cdot \vec{D} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_i C_i \sigma_j D_j \\
&= \sum_{i=1}^{3} \sigma_i^2 C_i D_i + \sum_{i<j} \left(\sigma_i \sigma_j C_i D_j + \sigma_j \sigma_i C_j D_j\right) \\
&= \vec{C} \cdot \vec{D} + \sum_{i<j} i\sigma_k \left(C_i D_j - C_j D_i\right) \\
&= \vec{C} \cdot \vec{D} + \sum_{i<j} i\sigma_k \left(\vec{C} \times \vec{D}\right)_k \\
&= \vec{C} \cdot \vec{D} + i\sigma \cdot \vec{C} \times \vec{D}
\end{aligned}$$

Solution. (b)

$$\vec{\Pi} \times \vec{\Pi} = \left(\vec{p} - \frac{q}{c}\vec{A}\right) \times \left(\vec{p} - \frac{q}{c}\vec{A}\right) = \overset{0}{\vec{p} \times \vec{p}} - \frac{q}{c}\left(\vec{A} \times \vec{p} + \vec{p} \times \vec{A}\right) + \overset{0}{\frac{q^2}{c^2}\vec{A} \times \vec{A}}$$

Consider the k'th component. Adding the two non-zero terms gives:

$$\begin{aligned}
\left(\vec{p} \times \vec{A}\right)_k &= p_i A_j - p_j A_i \quad = \quad -i\hbar \tfrac{\partial A_j}{\partial x_i} + A_j p_i - \left[-i\hbar \tfrac{\partial A_i}{\partial x_j} - A_i p_j\right] \\
\left(\vec{A} \times \vec{p}\right)_k &= A_i p_j - A_j p_i
\end{aligned}$$

$$\text{Sum} = -i\hbar\left[\frac{\partial}{\partial x_i} A_j - \frac{\partial}{\partial x_j} A_i\right] = -i\hbar(\left[\vec{\nabla} \times \vec{A}\right])_k = -i\hbar \vec{B}_k$$

So:

$$\begin{aligned}
\vec{\Pi} \times \vec{\Pi} = & -\frac{q}{c}\left(\vec{A} \times \vec{p} + \vec{p} \times \vec{A}\right) \; = \; -\frac{q}{c}(-i\hbar)\vec{\nabla} \times \vec{A} \\
= & \; \frac{iq\hbar}{c}\vec{\nabla} \times \vec{A} \; = \; \frac{iq\hbar}{c}\vec{B}
\end{aligned}$$

4. Scattering from a Spherical Shell.
Consider a particle of mmass μ scattering from a δ-function spherical shell potential $V(r) = aV_o\delta(r - a)$.

(i) Calculate the scattering amplitude in the Born approximation. For $ka = 10\pi$ (a large value of this dimensioness parameter), sketch the angluar (θ) dependence. Do the same for $ka = \pi/2$. [One can also a surprising behavior at fixed angle versus ka.]

(ii) Now, perform a partial wave analysis to find the $\ell = 0$ phase shift δ_o for $ka << 1$, where this phase shift should dominate.

Solution. See separate posting.