HW #6

4. Dirac Electron in Constant Magnetic Field -

Constant $\underset{\sim}{B}$ field: $\underset{\sim}{A}(\underset{\sim}{r}) = \frac{B}{2}(-y\hat{i}, x\hat{j}, 0)$.

Let's see what (20.2.3) looks like:

$[c\underset{\sim}{\alpha}\cdot\underset{\sim}{\pi} + \beta mc^2]\psi = E\psi$. Write in $2\otimes 2$ form

$$\left\{ c\begin{bmatrix} 0 & \underset{\sim}{\sigma} \\ \underset{\sim}{\sigma} & 0 \end{bmatrix}\cdot\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right) + \begin{bmatrix} I^{(2)} & 0 \\ 0 & -I^{(2)} \end{bmatrix} mc^2 \right\} \begin{bmatrix} \chi \\ \Phi \end{bmatrix} = E\begin{bmatrix} \chi \\ \Phi \end{bmatrix}$$

so

$$c\,\underset{\sim}{\sigma}\cdot\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)\Phi + mc^2\chi = E\chi$$

$$c\,\underset{\sim}{\sigma}\cdot\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)\chi - mc^2\Phi = E\Phi$$

$$c\,\sigma\cdot\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)\chi = (E + mc^2)\Phi \quad \text{so}$$

$$\boxed{\Phi = \frac{1}{E+mc^2}\, c\,\underset{\sim}{\sigma}\cdot\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)\chi.}$$ Subst. into 1^{st} Eqn:

$$c\,\underset{\sim}{\sigma}\cdot\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)\underbrace{\frac{1}{E+mc^2}}_{\text{scalar, multiply through}} c\,\underset{\sim}{\sigma}\cdot\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)\chi + \underbrace{mc^2\chi}_{\text{move}} = E\psi$$

$$c^2\left[\sigma\cdot\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)\right]^2\chi = (E + mc^2)(E - mc^2)\chi = (E^2 - m^2c^4)\chi$$

Now $\underset{\sim}{\sigma}$ commutes with functions, i.e. $\underset{\sim}{P}$ & $\underset{\sim}{A}$, so we need

$$\left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)^2 = \underset{\sim}{P}^2 - \frac{q}{c}\left[\underset{\sim}{P}\cdot\underset{\sim}{A} + \underset{\sim}{A}\cdot\underset{\sim}{P}\right] + \frac{q^2}{c^2}A^2. \quad A^2 = \left(\frac{B}{2}\right)^2(x^2+y^2) = \frac{B^2}{4}\rho^2$$

$$P_xA_x = -i\hbar\nabla_xA_x = -i\hbar\left[\overset{\to 0}{\cancel{(\nabla_xA_x)}} + A_x\nabla_x\right] = -i\hbar A_x\nabla_x = A_xP_x$$

so

$$\boxed{\underset{\sim}{P}\cdot\underset{\sim}{A} + \underset{\sim}{A}\cdot\underset{\sim}{P}} = 2\underset{\sim}{A}\cdot\underset{\sim}{P} = 2\left[A_x(-i\hbar\nabla_x) + A_y(-i\hbar\nabla_y)\right]$$

$$= 2\frac{B}{2}(-yP_x + xP_y) = \boxed{BL_z}, \quad L_z = xP_y - yP_x$$

From Ex. 12.3.8, given $\underset{\sim}{A} = \frac{B}{2}(-y, x, 0)$:

$\omega_0 = \frac{qB}{mc}$, and a canonical transformation gives the eigenvalues of

$$H\left(\frac{\omega_0}{2}\right) - \frac{\omega_0}{2} L_z \qquad -\frac{\omega_0}{2}\hbar M$$

$$= \left(K + \tfrac{1}{2}|M| - \tfrac{1}{2}M + \tfrac{1}{2}\right)\hbar\omega_0 - \frac{\omega_0}{2} L_z$$

K is any non-negative integer, M = ang. momentum Q number

Also, from Eq. 20.2.15 & following

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{\pi}\;\underset{\sim}{\sigma}\cdot\underset{\sim}{\pi} = \underset{\sim}{\pi}\cdot\underset{\sim}{\pi} + i\,\underset{\sim}{\sigma}\cdot(\underset{\sim}{\pi}\times\underset{\sim}{\pi})$$

$$= \left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right)^2 - \frac{q\hbar}{c}\underset{\sim}{\sigma}\cdot\underset{\sim}{B}$$

$-\frac{2q}{c}\underset{\sim}{S}\cdot\underset{\sim}{B}$, $\underset{\sim}{S} = \frac{\hbar}{2}\underset{\sim}{\sigma}$

w/ eigvals $\mp\frac{2q\hbar}{2c} = \mp\frac{q\hbar}{c}$

So, putting together

$$\left[E^2 - m^2c^4\right]\chi = c^2\left[\hbar^2k^2 + 2m\left\{K + \tfrac{1}{2}|M| - M + \tfrac{1}{2}\right\}\hbar\omega_0 - \frac{q\hbar}{c}\underset{\sim}{\sigma}\cdot\underset{\sim}{B}\right]\chi$$

Move m^2c^2 to r.h. side & factor it out $\frac{q\hbar c}{mc^2}$

$$E^2 = m^2c^4\left\{1 + \frac{\hbar^2k^2}{m^2c^2} + 2\frac{\hbar\omega_0}{mc^2}\left[K + \tfrac{1}{2}|M| - M + \tfrac{1}{2}\right] - \frac{\hbar q}{mc}\underset{\sim}{\sigma}\cdot\underset{\sim}{B}\right\}\chi$$

The exact energies are given by the square root of r.h. side.

This means both positive and negative energies. Note that E^2 already has two values, one for spin up $\underset{\sim}{\sigma}\cdot\underset{\sim}{B} > 0$, & down $\underset{\sim}{\sigma}\cdot\underset{\sim}{B} < 0$.

So there are four eigenvalues (it was a 4x4 matrix, after all.

The spinor parts of χ are arbitrary (but orthogonal),

we choose $\begin{bmatrix}1\\0\end{bmatrix}$ for up, $\begin{bmatrix}0\\1\end{bmatrix}$ for down.

OK, one thing not written explicitly above: we have taken

$$\chi = R(\rho)\, Q(z)\, F(\phi) \text{ x spinor}, \quad \rho = \sqrt{x^2+y^2}$$

$$Q(z) = e^{ikz}, \quad F(\phi) = e^{iM\phi}$$

To retrieve the full 4-component spinor, obtain Φ from

$$\Phi = \frac{1}{E+mc^2}\, c\, \underset{\sim}{\sigma} \cdot \left(\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A}\right) \chi .$$

That could be "interesting" to evaluate!

Let's look at the low energy regime, all energies small compared to mc^2.

$$E = mc^2 \left\{ 1 + \frac{\hbar^2 k^2}{m^2c^2} + 2\frac{\hbar\omega_0}{mc^2}\left[\,\cdots\cdots\,\right] - \frac{q\hbar}{m^2c^3}\, \underset{\sim}{\sigma}\cdot\underset{\sim}{B} \right\}^{1/2}$$

$$\simeq mc^2 + \frac{\hbar^2 k^2}{2m} + \hbar\omega_0\left[K + \tfrac{1}{2}|M| - M + \tfrac{1}{2}\right] - \frac{q\hbar}{2mc}\, \underset{\sim}{\sigma}\cdot\underset{\sim}{B} + \text{higher} \ldots$$

$$\left\{ \mu_B = \frac{e\hbar}{2mc} \right\}$$

Note: the $\frac{1}{2}$ exponent brings in necessary factors of two to make the non-relativistic energies as we know them.

(ii) The $B \to 0$ limit: $\omega_0 \to 0$ also.

$$E = \pm\sqrt{m^2c^4 + \hbar^2 k^2 c^2}, \text{ each doubly degenerate (spin)}$$

EXTRA

Consider the LARGE k regime: $\hbar k \gg mc$:

$E = \pm\, \hbar k c \left[1 + \frac{m^2}{2(\hbar k)^2} + \cdots \right]$. Huh? The electron thinks it has become a $\boxed{\text{photon}}$ $E \approx \hbar k c$. Of course, this theory is not the whole story at such high energy.