PHY 215 C MidTerm #1

#1 (i) $\delta(t)$ perturbation $H'(t) = \mathcal{U}\,\delta(t)$, $\mathcal{U}$ an operator.

This is an instantaneous "kick". The δ-fn gives a finite time integral. Thus there will be transition amplitude $i \to f$ of

$$\Delta d_f(0^+) = -\frac{i}{\hbar} \mathcal{U}_{fi} \int_{-\varepsilon}^{\varepsilon} dt'\, e^{i\omega_{fi}t'} \delta(t')\,dt' \; d_i(0^-)$$

$$= -\frac{i}{\hbar} \mathcal{U}_{fi}\, d_i(0^-)$$

& a transition probability of the absolute value squared.

(ii) Adiabatic. This perturbation is turned on and changed at a rate much slower than any intrinsic time scale of the system. Hence it will not undergo any transition to another state. The progression is

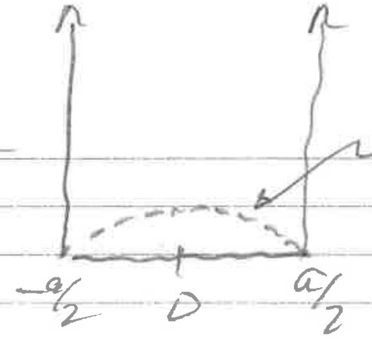
$$|\psi_j(t \to -\infty)\rangle \xrightarrow{\text{go to}} |\psi_j(t \to +\infty)\rangle.$$

Assumptions: the state j must be distinguishable from other states throughout:

- no degeneracies

- no level crossings during the (slow) time evolution.

#2. Square Well



perturb'n $V \cos(\pi \frac{x}{a}) e^{-i\omega t} \times \Theta(t)$.

turn on.

(i) What transitions are possible?

$$H'_{fi} \propto \int_{-a/2}^{a/2} \psi_f^*(x) \cos \pi \frac{x}{a}\, \psi_i(x)\, dx. \quad \psi_i \propto \cos(\cdots x) \text{ for odd } i = 1, 3, 5, \ldots$$

$$\psi_i \propto \sin(\cdots x) \text{ for even } i = 2, 4, 6, \ldots$$

$\cos(\cdots x)$ is even. $\psi_f^* \psi_i$ will be either even or odd.

If both f & i are even, $\psi_f^* \psi_i \sim \sin(\ )\sin(\ )$
$\sim$ odd $\times$ odd $=$ even

if f & i are odd, $\psi_f^* \psi_i \sim$ even $\times$ even $=$ even

Thus: selection rules require i & f to be <u>both even</u> <u>or</u> <u>both odd</u>.

(ii)
$$d_f(t) = -\frac{i}{\hbar} H'_{fi} \int_0^t dt'\, e^{i\omega_{fi} t'} e^{-i\omega t'} dt'\, d_i(0^-)$$

$$= -\frac{i}{\hbar} H'_{fi} \frac{1}{i(\omega_{fi}-\omega)} \left[e^{i(\omega_{fi}-\omega)t} - 1 \right] = -\frac{H'_{fi}}{\hbar} \frac{1}{\omega_{fi}-\omega} e^{i(\omega_{fi}-\omega)\frac{t}{2}}\, 2i\sin\left[(\omega_{fi}-\omega)t\right] \times d_i(0^-)$$

so
$$P_{i\to f}(t) = \left| \frac{2H'_{fi}}{\hbar} \frac{\sin(\omega_{fi}-\omega)t}{\omega_{fi}-\omega} \right|^2$$

(iii) If $E_f > E_i$, $\omega_{fi} > 0$ & there is a resonant denominator $(\omega_{fi}-\omega)^{-2}$ that diverges as $\omega \to \omega_{fi}$. Signals the system would like to conserve energy: $\hbar\omega = E_f - E_i$.

If $E_f < E_i$, $\omega_{fi} < 0$ and $(\omega_{fi}-\omega)^{-2}$ never diverges.

A $e^{-i\omega t}$ perturbation does not stimulate de-excitation, allow a little.

An extra observation: at small t, $P_{i-f}(t) \propto t^2$.

Ideal solution. Here Shankar notation $V(t) \to H'(t)$ is used.

#3. For interaction representation, the transformation freezes out the simple Schrödinger time "evolution":

$$|\psi_I(t)\rangle = \mathcal{U}_S^0(t,t_0)^\dagger |\psi_S(t)\rangle, \quad \mathcal{U}_S^0(t,t_0) = e^{-iH_S^0(t-t_0)/\hbar}$$

By construction $|\psi_I(t_0)\rangle = |\psi_S(t_0)\rangle$. Then calc. t-derivative

$$i\hbar \frac{d}{dt}|\psi_I(t)\rangle = i\hbar\left(\frac{\partial \mathcal{U}_S^{0\dagger}}{\partial t}\right)|\psi_S\rangle + \mathcal{U}_S^{0\dagger}\frac{d|\psi_S\rangle}{dt}$$

$$= -\mathcal{U}_S^{0\dagger} H_S^0 |\psi_S\rangle + \mathcal{U}_S^{0\dagger}(H_S^0 + H_S')|\psi_S\rangle$$

$$= \mathcal{U}_S^{0\dagger} H_S' |\psi_S\rangle = \mathcal{U}_S^{0\dagger} H_S' \mathcal{U}_S^0 \, \mathcal{U}_S^{0\dagger}|\psi_S\rangle$$

$$= \underbrace{\left(\mathcal{U}_S^{0\dagger} H_S' \mathcal{U}_S^0\right)}|\psi_I(t)\rangle \equiv H_I'(t)\,|\psi_I(t)\rangle$$

i.e. the "Sch. Eqn" for ψ_I is $\quad i\hbar\frac{d}{dt}|\psi_I(t)\rangle = H_I'(t)\,|\psi_I(t)\rangle.$

The "effective Hamiltonian" is $H_I'(t)$, the H^0-transformed $H'(t)$.

"t-dep. of ψ_I is determined by $H_I'(t)$", H^0 is hidden.

#4

At $t>0$: $H_0 = P^2/2\mu$. At $t>0$, $H_1 = (\underset{\sim}{P} - \frac{q}{c}\underset{\sim}{A})^2 - \underset{\sim}{\mu_s}\cdot\underset{\sim}{B}$.

$\underset{\sim}{\mu_s} = g_s \mu_B \underset{\sim}{S} = g_s \mu_B (\frac{1}{2}\underset{\sim}{\sigma})$ is the spin magnetic moment operator,

and $\underset{\sim}{B} = \underset{\sim}{\nabla}\times\underset{\sim}{A}$ is given. The perturbation is the difference

$$H'(t) = H_1(t) - H_0 = \left[\frac{-q}{2\mu c}(\underset{\sim}{P}\cdot\underset{\sim}{A} + \underset{\sim}{A}\cdot\underset{\sim}{P}) + \frac{q^2}{2\mu c^2}\underset{\sim}{A}^2 - \underset{\sim}{\mu_s}\cdot\underset{\sim}{B}\right]$$
$$\times\ \Theta(t) \leftarrow \text{turn on at } t=0.$$

Also, $g_s = 2$.
There will be a change in both orbital and spin states.

(ii) Neglect effect on orbital motion, then $H'(t) = \tilde{B}\sigma_z\,\Theta(t)$
where all constants are absorbed in $\boxed{\tilde{B} = \frac{1}{2}g_s\mu_B B = \mu_B B}$
1st order P. T. gives

$$d_j(t) = d_j(0) - \frac{i}{\hbar}\sum_{j'\neq j}\tilde{B}\int_0^t dt'\, e^{i\omega_{jj'}t'}\sigma_{z,jj'}\, d_{j'}(0)$$

(iii) A good guess is $|\psi_{initial}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, readily verified: $\langle\sigma_x\rangle = 1$, $\langle\sigma_y\rangle = 0 = \langle\sigma_z\rangle$

Thus $d_{up}(0) = \frac{1}{\sqrt{2}} = d_{dn}$. Also $\sigma_z|\psi_i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ so $\sum_{j'}\sigma_{z,jj'}d_{j'}(0)$
$= \frac{1}{\sqrt{2}}, j=up;\ -\frac{1}{\sqrt{2}}, j=dn$

So, $\sigma_z\begin{bmatrix}d_{up}\\d_{dn}\end{bmatrix} = \begin{bmatrix}d_{up}\\-d_{dn}\end{bmatrix}$. Now, sum on j' needs $\omega_{jj'}$ factor

$\omega_{jj'} = (E_j^0 - E_{j'}^0)/\hbar = 0$ if $j=j'$ (not needed)

$= \tilde{B} - (-\tilde{B}) = 2\tilde{B}$ if $j=up, j'=dn$

$= -2\tilde{B}$ if $j=dn, j'=up$

Then $d_{up}(t) = d_{up}(0) - \frac{i\tilde{B}}{\hbar}\int_0^t dt'\, e^{i2\tilde{B}t'/\hbar}\, d_{dn}(0)$

$$= d_{up}(0) - \frac{i\tilde{B}}{\hbar}\frac{\hbar}{i2\tilde{B}}\left(e^{i2\tilde{B}t/\hbar} - 1\right)\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left[1 - \frac{1}{2}e^{i\tilde{B}t/\hbar}\, 2i\sin\tilde{B}t\right]$$

$$= \frac{1}{\sqrt{2}}\left[1 - ie^{i\omega_0 t}\sin\omega_0 t\right]$$

{ $\omega_0 \equiv \tilde{B}/\hbar$ is called the cyclotron frequency }

Eqn for $d_{dn}(t)$ is similar, but with $d_{dn}(0) \to -d_{up}(0)$, and $2\tilde{B} \to -2\tilde{B}$ in the exponential, i.e. $\tilde{B} \to -\tilde{B}$ or $\omega_0 \to -\omega_0$ in final expression

$$d_{dn}(t) = \frac{1}{\sqrt{2}}\left[1 + ie^{-i\omega_0 t}\sin\omega_0 t\right]$$

The state is undergoing some sort of oscillation with frequency $\omega_0 = \mu_B B/\hbar$.

#5. Fermi Golden Rule.

(i) It is $R_{i\to f} = \frac{P_{i\to f}}{T} = \frac{2\pi}{\hbar} |H'_{fi}|^2 \delta(E_f^0 - E_i^0 - \hbar\omega)$.

- $R_{i\to f}$ is the time rate of transition $i \to f$.
- H'_{fi} is the usual transition matrix element (w/o t-dep.)
- δ-fn enforces conservation of energy.

Condition: perturb'n is periodic w/ single Fourier component $e^{-i\omega t}$

(ii) 2 δ-fn $\delta(\omega_{fi}-\omega)\,\delta(\omega_{fi}-\omega)$ arose.

One was represented as $\frac{1}{2\pi}\int_{-T/2}^{T/2} e^{-i(\omega_{fi}-\omega)t}\,dt$, $T\to\infty$.

Then the other used to set $\omega \equiv \omega_{fi}$, giving $\frac{T}{2\pi}$ with $T \to \infty$.

However, converting the probability to a rate involves

$R_{i\to} = \frac{P_{i-f}}{T}$ = probability per unit time

so T cancels out in the rate of transition,

which is what is observed anyway.