4. Consider the spherical shell $V(r) = aV_0\,\delta(r-a)$

a) Calculate the scattering amplitude in the Born approximation for $ka = 10\pi$. Sketch θ dependence, $ka = \pi/2$

$$f(\theta,\phi) = \frac{-\mu}{2\pi\hbar^2}\int e^{-i\vec{q}\cdot\vec{r}\,'}\,V(\vec{r}\,')\,d^3\vec{r}\,'$$

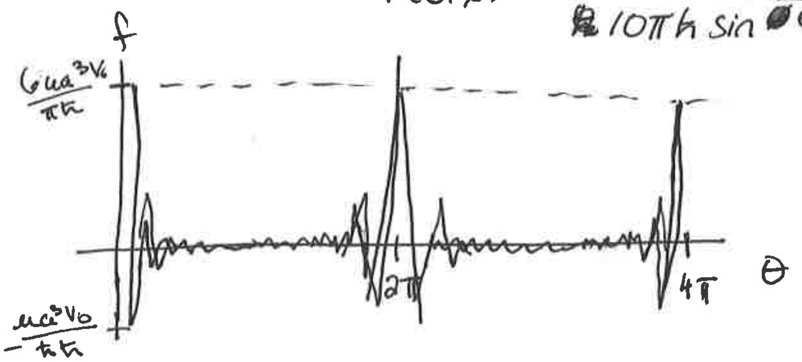
$$= -\frac{2\mu a V_0}{\hbar^2 q}\int r'\sin(qr')\,\delta(r'-a)\,dr'$$

$$= -\frac{2\mu a V_0}{\hbar^2 q}\,a\sin(qa) = -\frac{2\mu a^2 V_0}{\hbar q}\sin(qa)$$

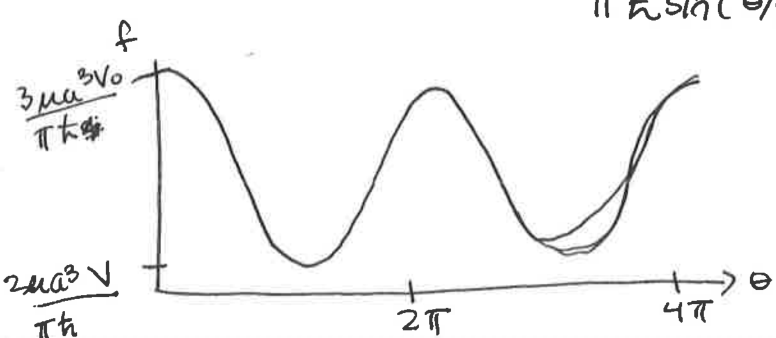
recall $q = 2k\sin\theta/2$

$$f(\theta,\phi) = -\frac{\cancel{2}\,\mu a^2 V_0}{\hbar k\sin\theta/2}\sin(2ka\sin\theta/2)$$

let $ak = 10\pi$: $f(\theta,\phi) = \dfrac{\mu a^3 V_0}{\cancel{2}\,10\pi\hbar\sin\theta/2}\sin(20\pi\sin\theta/2)$



let $ak = \pi/2$ $f(\theta,\phi) = \dfrac{2\mu a^3 V_0}{\pi\hbar\sin(\theta/2)}\sin\left(\frac{\pi}{2}\sin\theta/2\right)$



b) Now perform partial wave analysis

writing the general solution with hankel functions (from griffiths)

$$\psi(r,\theta) = \left\{ e^{ikz} + k \sum i^{\ell+1} (2\ell+1) a_\ell h_\ell^{(1)}(kr) P_\ell(\cos\theta) \right\} A$$

$$k^2 = 2mE/\hbar^2$$

$$\psi(r,\theta) = A \sum_\ell i^\ell (2\ell+1) \left[j_\ell(kr) + ik\, a_\ell h_\ell(kr) \right] P_\ell(\cos\theta)$$

we only need to worry about the $\ell = 0$ term

$$\psi_{ext} = A \left[j_0(kr) + ik\, a_0 h_0(kr) P_0(\cos\theta) \right]$$

asymptotic forms of the Bessel & Hankel

$$j_0(kr) \sim \sin kr / kr \qquad h_0(kr) \sim -i e^{ikr}/kr$$

$$\psi_{ext} = \frac{A}{kr} \left[\sin kr + k a_0 e^{ikr} \right]$$

inside we have the radial equation solution $u(r) = B \sin kr + D \cos kr$

$$R(r) = u/r = B \frac{\sin kr}{r} + D \cancel{\frac{\cos kr}{r}} \rightarrow \text{not finite at } r \to 0$$

must be continuous at the boundary

$$\frac{A}{ka}\left(\sin ka + k a_0 e^{ika} \right) = B \frac{\sin ka}{a}$$

we can't take an easy derivative due to the delta function

the radial equation is:

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left(V + \frac{\hbar^2}{2m} \frac{\ell(\ell+1)}{r^2} \right) u = Eu \implies -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + aV_0\, \delta(r-a) u = Eu$$

integrate across the boundery...

$$-\frac{\hbar^2}{2m}\int_{a-\epsilon}^{a+\epsilon}\frac{d^2u}{dr^2}\,dr + aV_0\int_{a-\epsilon}^{a+\epsilon}\delta(r-a)\,u\,dr = E\int_{a-\epsilon}^{a+\epsilon}u\,dr$$

as $\epsilon \to 0$ the RHS $\to 0$

$$\lim_{\epsilon\to 0} + \frac{\hbar^2}{2m}\int_{a-\epsilon}^{a+\epsilon}\frac{d^2u}{dr^2}\,dr = aV_0\,u(a)$$

$$\lim_{\epsilon\to 0}\frac{du}{dr}\Big|_{a-\epsilon}^{a+\epsilon} = \frac{2maV_0}{\hbar^2}u(a)$$

$$\left.\begin{array}{l} u_{ext} = \psi_{ext}\, r \\ u_{ext} = \frac{A}{k}\left(\sin kr + k a_0 e^{ikr}\right) \\ u_{in} = B\sin kr \end{array}\right.$$

$$\lim_{\epsilon\to 0}\frac{du}{dr}\Big|_{a-\epsilon}^{a+\epsilon} = \frac{du_{ext}}{dr}\Big|_{r=a} - \frac{du_{in}}{dr}\Big|_{r=a}$$

$$= A\cos(ka) + Aika_0 e^{ika} - Bk\cos ka$$

$$= (A-Bk)\cos ka + Aika_0 e^{ika}$$

$$(A-Bk)\cos ka + Aika_0 e^{ika} = \frac{2maV_0}{\hbar^2}B\sin ka$$

$$A(\sin ka + ka_0 e^{ika}) = Bk\sin ka$$

$$A\left(\cos ka + ika_0 e^{ika}\right) = B\left(k\cos ka + \frac{2maV_0}{\hbar^2}\sin ka\right)$$

$$\frac{B}{A} = \frac{\sin ka + ka_0 e^{ika}}{k\sin ka}$$

let $\beta = 2mV_0a^2/\hbar^2$

$$ika_0 e^{ika} = \frac{B}{A}\left(k\cos ka + \frac{\beta}{a}\sin ka\right) - \cos ka$$

$$= \left(\frac{1}{k} + \frac{a_0 e^{ika}}{\sin ka}\right)\left(k\cos ka + \frac{\beta}{a}\sin ka\right) - \cos ka$$

$$i k a_0 e^{ika} = \cos ka + a_0 k e^{ika} \cot ka + \frac{\beta}{ka} \sin ka + \frac{\beta a_0 e^{ika}}{a} - \cos ka$$

$$a_0 \left(ik e^{ika} - k e^{ika} \cot ka - \frac{\beta e^{ika}}{a} \right) = \frac{\beta \sin ka}{ka}$$

$$a_0 = \frac{\beta e^{-ika} \sin ka}{ka(ik - k\cot ka - \beta/a)} = \frac{\beta e^{-ika} \sin^2 ka}{k(ik\, a \sin ka - ka \cos ka - \beta \sin ka)}$$

Small angle approx ; $e^{-ika} \simeq 1$ gives

$$a_0 = \frac{\beta (ka)^2}{k(ika(ka) - ka - \beta ka)} = \frac{\beta a}{ika - 1 - \beta} \simeq -\frac{\beta a}{1+\beta}$$

$$a_0 = \frac{e^{2i\delta_0} - 1}{2ik} = -\beta a / 1+\beta$$

$$a_0 = \left[\frac{\beta}{a(ik - k\cot ka - \beta/a)} \right] \frac{e^{-ika} \sin(ka)}{k} = \frac{e^{i\delta_e} \sin \delta_e}{k}$$

so pluging this into a solver

$$\left[\frac{\beta}{ika - ka \cot ka - \beta} \right] \sim \frac{-\beta}{(ka + \cot ka + \beta)}$$

$$\delta_0 = -ka + \tan^{-1} \left(\frac{-\beta}{ka \cot ka + \beta} \right)$$