Prob. 1

Example of Born Approximation: Square Well Potential

Consider scattering of particles interacting via a spherical three dimensional (3D) square well potential $V(r) = V_\circ$ for $r \le R_\circ$ and zero outside ($V(r) = 0$ for $r > R_\circ$). The integral (35) for the scattering amplitude required here is then

$$f_k^{(1)}(\theta) = \frac{2m}{\hbar^2}\frac{1}{Q}\int_0^{R_\circ} V_\circ r \sin Qr \, dr = \frac{2m}{\hbar^2}\frac{1}{Q}V_\circ \left[\frac{\sin Qr - Qr\cos Qr}{Q^2}\right]_0^{R_\circ} \tag{38}$$

[7] and whence to the differential cross section

$$\left(\frac{d\sigma}{d\theta}\right)^{(1)} = \left(\frac{2m}{\hbar^2}\frac{V_\circ}{Q}\right)^2 R_\circ^2 j_1^2(QR_\circ) \simeq \left(\frac{2m}{\hbar^2}\frac{V_\circ}{Q}\right)^2 \begin{cases} \frac{1}{9}(1-\frac{1}{5}Q^2R_\circ^2) & \text{for low } E, \quad kR_\circ < 1 \\ \frac{R_\circ^2}{Q^2} & \text{for high } E, \quad kR_\circ > 1 \end{cases} \tag{39}$$

From integrating over θ and ϕ the low and high energy limits for the total cross section are

$$\sigma(E\to\infty) = \pi\left(\frac{2m}{\hbar^2}\right)^2\left(\frac{V_\circ R_\circ^3}{kR_\circ}\right)^2 \qquad \sigma(E\to 0) = \sigma(E\to\infty)\frac{8}{9}\left(k^2R_\circ^2 - \frac{2}{5}k^4R_\circ^4 + \cdots\right). \tag{40}$$

The two examples illustrate some general features of scattering in the Born approximation:
(i) Born approximation is based on perturbation theory, so it works best for high energy particles.
(ii) At high energy, the scattering amplitude and the cross section are inversely proportional to the energy ($E = \hbar^2k^2/2m$). E.g. both become smaller and the scattering weaker with increasing energy. This is a general phenomenon, if no bound states appear in the vicinity of the energy. This can be seen best by inspecting the Fourier transformed Green function $G_\circ(k|E) \propto 1/(E - \frac{\hbar^2k^2}{2m})$ that is inverse proportional to the energy.
(iii) Scattering depends on square of the interaction potential, e.g. $V_\circ^2$, so both attractive and repulsive potentials behave the same.
(iv) The dependence on the energy of the incident beam k and scattering angle θ arises only through the combination $Q = 2k\sin\frac{\theta}{2}$. Thus as energy increases, the scattering angle θ is reduced and the scattered beam becomes more peaked in the forward direction.
(v) Angular dependence depends on the range of the potential $R_\circ$ but not on the strength $V_\circ$.
(vi) The total cross section depends on both range $R_\circ$ and depth $V_\circ$ of the potential.

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#2 i) Show that for a ~~the~~ 100 MeV kinetic energy neutron incident on a fixed nucleus, the maximum partial wave is $l_{max} = 2$. The range of the nuclear force is roughly one $F = 10^{-5}$ Å. Also $\hbar c = 200$ MeV F

ii) Repeat this with a Higgs Boson moving at $c/2$.

$r_0 = 10^{-5}$ Å $= 1F$

$E = 100$ MeV

$k = \sqrt{2E\mu}/\hbar$

$\mu = 938$ MeV$/c^2$

$\hbar c =$ ~~MeV F~~ 200 MeV F

from Shankar

$$l_{max} \simeq k r_0$$

$$l_{max} = \frac{r_0\sqrt{2E\mu}}{\hbar} = \frac{1F\sqrt{2(100\,MeV)(938\,MeV)}}{200\,MeV\,F}$$

$$l_{max} = 2.17 \simeq 2$$

ii) for the higgs boson

$\mu = 125$ GeV$/c^2$

$$E = \frac{1}{2}\mu v^2 = \frac{1}{2}(125\,GeV/c^2)\left(\frac{c^2}{4}\right)$$

$$E = \frac{125}{8}\ GeV$$

GeV

$$l_{max} = \frac{1F\sqrt{2(125/8\,GeV)(125\,GeV)}}{200\,MeV\,F} = 0.3125 \times 10^3 \rightarrow l_{max} \simeq 312$$

(!)

3

The hard sphere potential is defined by Shanker as $V(r)$ is ∞ for $r<r_0$ and zero outside. Show that $\sigma_{\ell=0}(k\to 0) = 4\pi r_0^2$. Why is the effective radius $2r_0$ rather than r_0?

Shankar goes through the partial wave expansion starting on page 545

we know that in the sphere $R_\ell(r)$ vanishes, ~~exists~~ and outside it is given by

$$R_\ell(r) = A_\ell j_\ell(kr) + B_\ell n_\ell(kr)$$

$$R_\ell(r) \xrightarrow[r\to\infty]{} \frac{(A_\ell^2+B_\ell^2)^{1/2}}{kr}\left[\sin\left(kr - \frac{\ell\pi}{2} + \delta_\ell\right)\right]$$

with $\delta_\ell = \tan^{-1}\left[\frac{j_\ell(kr_0)}{n_\ell(kr_0)}\right]$

for $\ell = 0$

$$\delta_0 = -\tan^{-1}(\tan(kr_0)) = -kr_0$$

from 19.5.17

$$f(\theta) = \frac{1}{k}\sum_\ell (2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos\theta)$$

let $\ell = 0$

$$f(\theta) = \frac{1}{k}e^{-ikr_0}\sin(kr_0)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\sin^2(kr_0)}{k^2}$$

since $k \to 0$ use the small angle approximation

$$\frac{d\sigma}{d\Omega} \to \frac{(kr_0)^2}{k^2} = r_0^2$$

then: $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi r_0^2$

this is 4 x the geometric crosssection πa^2. low energy scattering implies long wavelengths $\rightarrow$ this could result in the sphere appearing much ~~larger~~ larger than the actual size of the potential. Good observation

4. $\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$ (19.5.18)

Starting with

$\sigma = \int |f|^2 d\Omega$

and

$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)$

$\Rightarrow \sigma = \frac{1}{k^2}\int \left|\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)\right|^2 d\Omega$

Since $P_l P_m = \delta_{l,m}$ $\frac{2}{2l+1}$

$\sigma = \frac{1}{k^2}\sum_{l=0}^{\infty}(2l+1)^2 e^{i\delta_l - i\delta_l}\sin^2\delta_l \int P_l^2(\cos\theta)\,d\Omega$

$= \frac{1}{k^2}\sum_{l=0}^{\infty}(2l+1)^2 \sin^2\delta_l \left(\frac{2}{2l+1}\right)(2\pi)$

$\boxed{= \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l}$

Using $P_l(\cos(0)) = 1$

$f(0) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)(\cos\delta_l + i\sin\delta_l)\sin\delta_l$

$\mathrm{Im}(f(0)) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$

$\Rightarrow \boxed{\sigma = \frac{4\pi}{k}\mathrm{Im}(f(0))}$