HW #4 Solutions

Prob. 1. The expression for $d_f^{(2)}(t)$, assuming $e^{-i\omega t}$ driving term, is given on the following two pages, with a little extra (transition rate).

About the matrix elements ~~of~~ of X in our H.O. system (frequency Ω), we know that (interpreting F, M, I is the integer denoting the state) $M = I \pm 1$, and
$F = M \pm 1 = (I \pm 1) \pm 1$. Thus F can be $I-2$, I, or $I+2$. The sum on M has only two terms, with $\omega_{mi} = \pm\Omega$.

Thus there are resonances at $\omega = \pm\Omega$.

The accompanying sheets indicate there transitions finally only to
$F = I \pm 2$ (energy conserving).

Harmonic perturbations: second order transitions

- Although first order perturbation theory often sufficient, sometimes $\langle \mathrm{f}|V|\mathrm{i}\rangle = 0$ by symmetry (e.g. parity, selection rules, etc.). In such cases, transition may be accomplished by indirect route through other non-zero matrix elements.

- At second order of perturbation theory,

$$c_{\mathrm{f}}^{(2)}(t) = -\frac{1}{\hbar^2}\sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, e^{i\omega_{\mathrm{f}m}t' + i\omega_{m\mathrm{i}}t''} V_{\mathrm{f}m}(t')V_{m\mathrm{i}}(t'')$$

- If harmonic potential perturbation is gradually switched on, $\boxed{V(t) = e^{\varepsilon t}\, Ve^{-i\omega t},\ \varepsilon \to 0,}$ with the initial time $t_0 \to -\infty$,

$$c_{\mathrm{f}}^{(2)}(t) = -\frac{1}{\hbar^2}\sum_m \langle \mathrm{f}|V|m\rangle\langle m|V|\mathrm{i}\rangle$$

$$\times \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt''\, e^{i(\omega_{\mathrm{f}m} - \omega - i\varepsilon)t'} e^{i(\omega_{m\mathrm{i}} - \omega - i\varepsilon)t''}$$

Harmonic perturbations: second-order transitions

- From time integral,

$$c_{\cancel{i}f}^{(2)} = -\frac{1}{\hbar^2} e^{i(\omega_{\rm fi}-2\omega)t} \frac{e^{2\varepsilon t}}{\omega_{\rm fi} - 2\omega - 2i\varepsilon} \sum_m \frac{\langle \mathrm{f}|V|m\rangle\langle m|V|\mathrm{i}\rangle}{\omega_{mi} - \omega - i\varepsilon}$$

EXTRA

- Leads to transition rate ($\varepsilon \to 0$): Work this out for yourself, to see how it works.

$$\boxed{\frac{d}{dt}|c_{\cancel{i}f}^{(2)}(t)|^2 = \frac{2\pi}{\hbar^4} \left| \sum_m \frac{\langle \mathrm{f}|V|m\rangle\langle m|V|\mathrm{i}\rangle}{\omega_{mi} - \omega - i\varepsilon} \right|^2 \delta(\omega_{\rm fi} - 2\omega)}$$

This looks like a generalized Fermi Golden Rule.

- This translates to a transition in which system gains energy $2\hbar\omega$ from harmonic perturbation, i.e. two "photons" are absorbed – Physically, first photon takes effects virtual transition to short-lived intermediate state with energy ω_m.
- If an atom in an arbitrary state is exposed to monochromatic light, other second order processes in which two photons are emitted, or one is absorbed and one emitted are also possible.

Prob. 2. The (a?) natural energy scale is $\hbar^2/\mu r_0^2 \equiv E_0$.

The result is

$$\sigma_\gamma = 16\pi r_0^2 \frac{g^2}{E_0^2} \frac{1}{1+4k^2 r_0^2}. \qquad k^2 r_0^2 = \frac{\hbar^2 k^2}{2\mu}\cdot\frac{2\mu r_0^2}{\hbar^2} = \frac{E_k}{\frac{1}{2}E_0} = 2\frac{E_k}{E_0}$$

$$\{E_k \to E\} = 16\pi r_0^2 \frac{g^2}{E_0^2} \frac{1}{1+8\,E/E_0} \quad E/E_0$$

$$= 16\pi r_0 \frac{g^2}{E_0^2} \frac{E_0}{E_0+8E}$$ ← maybe most natural way to write.

low E: set $E \to 0$, no problem, $\boxed{\sigma_\gamma \to 16\pi r_0^2 \frac{g^2}{E_0^2}}$ constant

high E: goes to zero as $\boxed{2\pi r_0^2 \frac{g^2}{E_0 E}}$

Prob. 3 Ex. 19.3.2. Some students plots will be chosen for display.