Problem 2.

Square well, $0<x<a$. $H'(t) = V\cos\frac{\pi x}{a}\sin\omega t\,\Theta(t)$

Reference system: $\psi_j^0(x) = \sqrt{\frac{2}{a}}\sin\left(j\pi\frac{x}{a}\right)$, $E_j^0 = j^2 E_{j=1}^0$

natural energy scale

Initial condition: $d_i(t<0)=1$
for some initial state i

1st order, $f\neq i$:

$$d_f(t) = \left(\frac{-i}{\hbar}\right)\int_0^t \langle f|H'|i\rangle e^{i\omega_{fi}t'}dt', \quad \omega_{fi} = \frac{E_f^0 - E_i^0}{\hbar}$$

$$\cancel{= \frac{-i}{\hbar}\langle f|H'|i\rangle}$$

$$= \frac{-i}{\hbar}H'_{fi}\frac{1}{2i}\int_0^t e^{i(\omega_{fi}+\omega)t'} - e^{i(\omega_{fi}-\omega)t'}dt', \quad H'_{fi} = \langle f|V\cos\frac{\pi x}{a}|i\rangle$$

$$= \frac{-H'_{fi}}{2\hbar}\left[\frac{e^{i(\omega_{fi}+\omega)t}}{i(\omega_{fi}+\omega)} - \frac{e^{i(\omega_{fi}-\omega)t}}{i(\omega_{fi}-\omega)} + \frac{1}{i(\omega_{fi}-\omega)} - \frac{1}{i(\omega_{fi}+\omega)}\right]$$

$$= \frac{-H'_{fi}}{\hbar(\omega_{fi}^2-\omega^2)}\left[e^{i\omega_{fi}t}(i\omega\cos\omega t + \omega_{fi}\sin\omega t) - i\omega\right]$$

On the other hand,

$$H'_{fi} = \langle f|H'|i\rangle = \langle f|V\cos\frac{\pi x}{a}|i\rangle = \frac{2V}{a}\int_0^a \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{f\pi x}{a}\right)\sin\left(\frac{i\pi x}{a}\right)dx$$

$$= \frac{V}{2a}\int_0^a\left[\cos\left(\frac{(1+(i-f))\pi x}{a}\right) + \cos\left(\frac{(1+(f-i))\pi x}{a}\right) - \cos\left(\frac{(1+(i+f))\pi x}{a}\right) - \cos\left(\frac{(1-(i+f))\pi x}{a}\right)\right]dx$$

$$= \begin{cases} V/2 & \text{only if } f = i\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\# \Rightarrow d_f(t) = \frac{-V}{2\hbar(\omega_{fi}^2 - \omega^2)}\left[e^{i\omega_{fi}t}(i\omega\cos\omega t + \omega_{fi}\sin(\omega t)) - i\omega\right]$$

if $f = i \pm 1$,

$$\Rightarrow P_f(t) = |d_f(t)|^2$$

$$= \begin{cases} \frac{V^2}{4\hbar^2(\omega_{fi}^2 - \omega^2)}\left| e^{i\omega_{fi}t}(i\omega\cos\omega t + \omega_{fi}\sin\omega t) - i\omega\right|^2 & \text{only if } f = i \pm 1 \\ 0 & \text{, otherwise} \end{cases}$$

$$= \begin{cases} \frac{V^2}{4\hbar^2(\omega_{fi}^2 - \omega^2)}\left[\omega^2\cos^2\omega t + \omega_{fi}^2\sin^2\omega t + \omega^2 - 2\omega^2\cos\omega t\cos\omega_{fi}t - 2\omega\omega_{fi}\sin\omega t\sin\omega_{fi}t\right] & \text{, only if } f = i \pm 1 \\ 0 & \end{cases}$$

3) i) spin 1/2 particle in field B_z, spin-up initially

Perturbed by $H'(t) = \sigma_x B_x \sin(\Omega t)\theta(t)$ $\quad (H^\circ = \sigma_z B_z) \rightarrow H'(t) = H'\left(\frac{e^{i\Omega t} - e^{-i\Omega t}}{2i}\right)\theta(t)$ if $H' = \sigma_x B_x$

Want to find probability particle shifts from $\begin{bmatrix}1\\0\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}$

$$d_d(t) = -\frac{B_x}{2\hbar}\int_0^t \langle d|\sigma_x|u\rangle (e^{i\Omega t} - e^{-i\Omega t}) e^{i\omega_{du} t}\, dt'$$

(down, up)

$E_r = B_z \quad E_\downarrow = -B_z$

$\rightarrow \omega_{du} = \frac{-B_z - B_z}{\hbar} = \frac{-2B_z}{\hbar}$

$$= -\frac{B_x}{2\hbar}\langle d|\sigma_x|u\rangle \int_0^t e^{i(\Omega + \omega_{du})t} - e^{i(-\Omega+\omega_{du})t}\, dt$$

$$\rightarrow d_d(t) = -\frac{B_x}{2\hbar}\langle d|\sigma_x|u\rangle \left[\frac{e^{i(\Omega+\omega_{du})t} - 1}{i(\Omega+\omega_{du})} - \frac{e^{i(-\Omega+\omega_{du})t} - 1}{i(-\Omega+\omega_{du})}\right] = -\frac{B_x \langle d|\sigma_x|u\rangle}{\hbar(\omega_{du}^2 - \Omega^2)}\left[e^{i\omega_{du}t}(i\Omega\cos(\Omega t) - \omega_{du}\sin(\Omega t)) - i\Omega\right]$$

see problem #2. The time dependence is the same

Now, $\langle d|\sigma_x|u\rangle = i[0\ 1]\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = i[0\ 1]\begin{bmatrix}0\\1\end{bmatrix} = 1$

$$\rightarrow d_\downarrow(t) = \frac{B_x}{\hbar(\omega_{du}^2 - \Omega^2)}\left[e^{i\omega_{du}t}(i\Omega\cos(\Omega t) - \omega_{du}\sin(\Omega t)) - i\Omega\right]$$

-1

$$\rightarrow P_\downarrow(t) = |d_\downarrow(t)|^2 = \frac{B_x^2/\hbar^2}{(\omega_{du}^2 - \Omega^2)^2}\left|e^{i\omega_{du}t}(i\Omega\cos(\Omega t) - \omega_{du}\sin(\Omega t)) - i\Omega\right|^2$$

$\leftarrow$ not easy to simplify further

The time dependence of the probability will be the same as in problem #2. Thus there will be a limit of sorts as $t \rightarrow \infty$, as the periodicity will carry over.

The Ω dependence will be on the order of Ω^{-2} $\Rightarrow$ as Ω gets very large, P will $\rightarrow 0$.

This makes sense, as a large Ω would essentially remove the perturbation, making it extremely gradual.

ii) Want to find $\langle \vec{S} \rangle$ and characterize the time dependence

$$\langle \vec{S} \rangle = (\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle)$$

$\langle S_z \rangle = \frac{\hbar}{2}$ if it's $\uparrow$ $\quad -\frac{\hbar}{2}$ if it's $\downarrow$

$\rightarrow \langle S_z \rangle = \frac{\hbar}{2}(\text{Probability } \uparrow) - \frac{\hbar}{2}(\text{Probability } \downarrow) = \frac{\hbar}{2}(1 - |d_\downarrow(t)|^2) - \frac{\hbar}{2}|d_\downarrow(t)|^2$

$\rightarrow \boxed{\langle S_z \rangle = \frac{\hbar}{2} - \hbar\, d_\downarrow^* d_\downarrow}$

Now, $\langle S_x \rangle = \langle \Psi(t) | S_x | \Psi(t) \rangle = \frac{1}{2}\langle \Psi(t) | S_+ + S_- | \Psi(t) \rangle$

$$|\Psi(t)\rangle = d_\downarrow(t) e^{-iE_\downarrow t/\hbar} |\downarrow\rangle + d_\uparrow e^{-iE_\uparrow t/\hbar} |\uparrow\rangle$$

$-B_z \quad B_z$

$$d_\uparrow = 1 - \frac{i}{\hbar}\int_0^t \langle \uparrow | H'(t') | \uparrow \rangle e^{i\omega_{\uparrow\uparrow} t'} dt'$$

$\rightarrow d_\uparrow = 1 \qquad 0$ as $\langle \uparrow | \sigma_x | \uparrow \rangle = 0$

$\rightarrow |\Psi(t)\rangle = d_\downarrow(t) e^{+iB_z t/\hbar} |\downarrow\rangle + d_\uparrow e^{-iB_z t/\hbar} |\uparrow\rangle$

$\rightarrow \frac{1}{2}\langle \Psi(t) | S_+ + S_- | \Psi(t) \rangle = \frac{1}{2}\left(\langle \downarrow | d_\downarrow(t)^* e^{-iB_z t/\hbar} + \langle \uparrow | d_\uparrow(t)^* e^{iB_z t/\hbar}\right)(S_+ + S_-)\left(d_\downarrow(t) e^{iB_z t/\hbar} |\downarrow\rangle + d_\uparrow(t) e^{-iB_z t/\hbar} |\uparrow\rangle\right)$

OK

$$= \frac{\hbar}{2}\left(d_\uparrow(t)^* d_\downarrow(t)\, e^{i2B_z t/\hbar} + d_\downarrow^*(t)\, d_\uparrow(t)\, e^{-2iB_z t/\hbar}\right)$$

$\rightarrow \boxed{\langle S_x \rangle = \frac{\hbar}{2} d_\downarrow(t) e^{i2B_z t/\hbar} + \frac{\hbar}{2} d_\downarrow^* e^{-i2B_z t/\hbar}}$

$\langle S_y \rangle = \langle \Psi(t) | S_y | \Psi(t) \rangle = \frac{1}{2i}\langle \Psi(t) | S_+ - S_- | \Psi(t) \rangle$

$\rightarrow \langle S_y \rangle = \frac{1}{2i}\left(\langle \downarrow | d_\downarrow^* e^{-iB_z t/\hbar} + \langle \uparrow | d_\uparrow^* e^{iB_z t/\hbar}\right)(S_+ - S_-)\left(d_\downarrow e^{iB_z t/\hbar} |\downarrow\rangle + d_\uparrow e^{-iB_z t/\hbar} |\uparrow\rangle\right)$

$\rightarrow \boxed{\langle S_y \rangle = \frac{1}{2i}\left(-d_\downarrow^* e^{-i2B_z t/\hbar} + d_\downarrow e^{i2B_z t/\hbar}\right)}$

It's difficult to characterize this time dependence of $\langle \vec{S} \rangle$. The expectation value of S_x will be imaginary, as $d_\downarrow$ contains factors of i, but S_y will not due to the $\frac{1}{i}$ factor present.
We will see that these expectation values are periodic (the exponential factors), which makes sense as we expect our particle to be oscillating between spin up and down.
It is odd that $\langle S_y \rangle$ will be periodic, as there are no measurements being made in the $\hat{y}$ direction.