Problem 1 (Wenjian)

1) H for a particle in the potentials $(\vec{A}, \phi)$:

$$H = \frac{1}{2\mu}\left(\hat{p} - \frac{q}{c}\vec{A}\right) + q\phi$$

where μ is the mass and $\hat{p}$ is the momentum operator and q is the charge.

2) Apply the gauge transformation:

$$\begin{cases} \vec{A}' = \vec{A} - \vec{\nabla}\Lambda \\ \phi' = \phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t} \end{cases}$$

where Λ is an arbitrary function.

We get:

$$H_\Lambda = \frac{1}{2\mu}\left(\hat{p} - \frac{q}{c}(\vec{A} - \vec{\nabla}\Lambda)\right)^2 + q\left(\phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}\right)$$

3) $\psi(r,t)$ satisfy $H\psi = i\hbar\frac{\partial \psi}{\partial t}$. In this question, we need to prove: $H_\Lambda \psi_\Lambda = i\hbar\frac{\partial \psi_\Lambda}{\partial t}$, $\psi_\Lambda = e^{-\frac{iq\Lambda}{\hbar c}}\psi$

① $$i\hbar\frac{\partial \psi_\Lambda}{\partial t} = i\hbar e^{-\frac{iq\Lambda}{\hbar c}}\frac{\partial \psi}{\partial t} + \frac{q}{c}\frac{\partial \Lambda}{\partial t}e^{-\frac{iq\Lambda}{\hbar c}}\psi$$

② $$H_\Lambda \psi_\Lambda = \frac{1}{2\mu}\left(\hat{p} - \frac{q}{c}(\vec{A} - \vec{\nabla}\Lambda)\right)^2\psi_\Lambda + q\phi e^{-\frac{iq\Lambda}{\hbar c}}\psi + \frac{q}{c}\frac{\partial \Lambda}{\partial t}e^{-\frac{iq\Lambda}{\hbar c}}\psi$$

Because $\frac{1}{2\mu}\left(\hat{p} - \frac{q}{c}(\vec{A} - \vec{\nabla}\Lambda)\right)^2\psi_\Lambda =$ ((continue)

$$= \frac{1}{2\mu}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla}\Lambda\right)\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla}\Lambda\right)e^{-\frac{iq\Lambda}{\hbar c}}\psi$$

$$= \frac{1}{2\mu}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla}\Lambda\right)e^{-\frac{iq\Lambda}{\hbar c}}\left(-i\hbar\vec{\nabla} + (-i\hbar)\left(-\frac{iq}{\hbar c}\right)\vec{\nabla}\Lambda - \frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla}\Lambda\right)\psi$$

$$= \frac{1}{2\mu}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla}\Lambda\right)e^{-\frac{iq\Lambda}{\hbar c}}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)\psi$$

$$= \frac{1}{2\mu}\, e^{-\frac{iq\Lambda}{\hbar c}}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)^2\psi$$

$$= e^{-\frac{iq\Lambda}{\hbar c}} \cdot \frac{1}{2\mu}\left(\hat{p} - \frac{q}{c}\vec{A}\right)^2\psi$$

Thus, we have

$$H_\Lambda \psi_\Lambda = e^{-\frac{iq\Lambda}{\hbar c}} \cdot \frac{1}{2\mu}\left(\hat{p} - \frac{q}{c}\vec{A}\right)^2\psi + e^{-\frac{iq\Lambda}{\hbar c}}(q\phi\psi) + \frac{q}{c}\frac{\partial\Lambda}{\partial t}e^{-\frac{iq\Lambda}{\hbar c}}\psi$$

$$= e^{-\frac{iq\Lambda}{\hbar c}} \cdot \left[\frac{1}{2\mu}\left(\hat{p} - \frac{q}{c}\vec{A}\right)^2\psi + q\phi\psi\right] + \frac{q}{c}\frac{\partial\Lambda}{\partial t}e^{-\frac{iq\Lambda}{\hbar c}}\psi$$

$$= e^{-\frac{iq\Lambda}{\hbar c}} \cdot i\hbar\frac{\partial\psi}{\partial t} + \frac{q}{c}\frac{\partial\Lambda}{\partial t}e^{-\frac{iq\Lambda}{\hbar c}}\psi = i\hbar\frac{\partial\psi_\Lambda}{\partial t}$$

$$\Rightarrow \quad H_\Lambda\psi_\Lambda = i\hbar\frac{\partial\psi_\Lambda}{\partial t}$$

Comment. (Discussion?) This does not prove gauge invariance. It proves that the derived Hamilton H_Λ & wvfn ψ_Λ also satisfy Sch. Eqn. But $H_\Lambda \neq H$, $\psi_\Lambda(\underset{\sim}{r},t) \neq \psi(\underset{\sim}{r},t)$, and it takes more work to show that they always lead to the identical measurable properties.

Prob. 3. Driven $S=1/2$ spin. $H^0+H'(t)=\sigma_z B_z+\sigma_x B_x \sin(\Omega t)\,\Theta(t)$.
This is a classic example of a "2-level system". Coupled to other 2LSs, to other degrees of freedom, or to a bath, 2LSs have been studied intensely.

H^0 Eigensystem. $H^0=\begin{bmatrix} B_z & 0 \\ 0 & -B_z\end{bmatrix}$.

$|up\rangle=\begin{bmatrix}1\\0\end{bmatrix}$, $\varepsilon_{up}=B_z$ (B_z might be of either sign)

$|dn\rangle=\begin{bmatrix}0\\1\end{bmatrix}$, $\varepsilon_{dn}=-B_z$

$H'(t)=H'F(t)$, $H'=B_x\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $F(t)=\sin\Omega t$ for $t>0$.

Initial condition: $d_{up}(t)=1$ for $t<0$.
$d_{dn}(t)=0$ " " .

1st order t-dep. P.T.

Note: H' has only off-diagonal matrix elements, so d_{up} will not change to 1st order $\boxed{H_{dn,up}=B_x}$

For $|dn\rangle$ component

$$d_{dn}(t)=\text{zero}-\frac{i}{\hbar}B_x\int^t \sin\Omega t'\, e^{i\omega t'}\,dt', \quad \omega\equiv(\varepsilon_{up}-\varepsilon_{dn})/\hbar = \omega_{fi}$$

$$=-\frac{i}{\hbar}B_x\int_0^t dt'\,\frac{e^{i\Omega t'}-e^{-i\Omega t'}}{2i}\,e^{i\omega t'}$$

$$=-\frac{B_x}{2\hbar}\left[\frac{e^{i(\omega+\Omega)t}-1}{i(\omega+\Omega)}-(\Omega\to-\Omega)\right]$$

$$=-\frac{B_x}{2i\hbar}\left[(\omega-\Omega)\left(e^{i(\omega+\Omega)t}-1\right)-(\omega+\Omega)\left(e^{i(\omega-\Omega)t}-1\right)\right]\frac{1}{\omega^2-\Omega^2}$$

$$=\frac{iB_x e^{i\omega t}}{2\hbar(\omega^2-\Omega^2)}\left[\omega\left(e^{i\Omega t}\cancel{-1}-e^{-i\Omega t}\cancel{+1}\right)-\Omega\left(e^{i\Omega t}-1+e^{-i\Omega t}-1\right)\right]$$

$$=\frac{iB_x e^{i\omega t}}{2\hbar(\omega^2-\Omega^2)}\left[2i\omega\sin\Omega t-2\Omega(\cos\Omega t-1)\right]$$

The form of $|d_{dn}(t)|^2$ is probably not very helpful to look at. PLOT

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Let's just look at the numerator at $\Omega = \omega$. Of course the denominator vanishes $\Rightarrow$ 1st order PT breaks down.

$\Omega = \omega$:

$$2i\omega \sin\omega t - 2\omega\cos\omega t + 2\omega = -2\omega\left[\cos\omega t - i\sin\omega t - 1\right]$$

$$= -2\omega\left[e^{-i\omega t} - 1\right] = -2\omega\, e^{-i\omega t/2}\left[e^{-i\omega t/2} - e^{+i\omega t/2}\right]$$

$$= -2\omega\, e^{-i\omega t/2}\left(-2i\sin(\omega t/2)\right) = 4i\omega\sin\frac{\omega t}{2}\, e^{-i\omega t/2}.$$

Behavior?

- increases linearly from zero for small $t > 0$
- then oscillates with frequency $\omega/2$ forever.

For the plot: $\omega \equiv \omega_{fi}$ is the natural freq'y of the system = energy scale

Plot $|d_{dn}(t)|^2$ vs. t for several values of Ω/ω:

shown in the next page

say, $\Omega/\omega = 1/4, 1/2, \cancel{1}, 2, 4$.

B_x/B_z is another parameter. $B_x/B_z = 0.1 \Rightarrow |B_x/B_z|^2 \approx 10^{-2}$, 1st order should be reasonable.

Part (2). The expectation value of the direction of the spin is

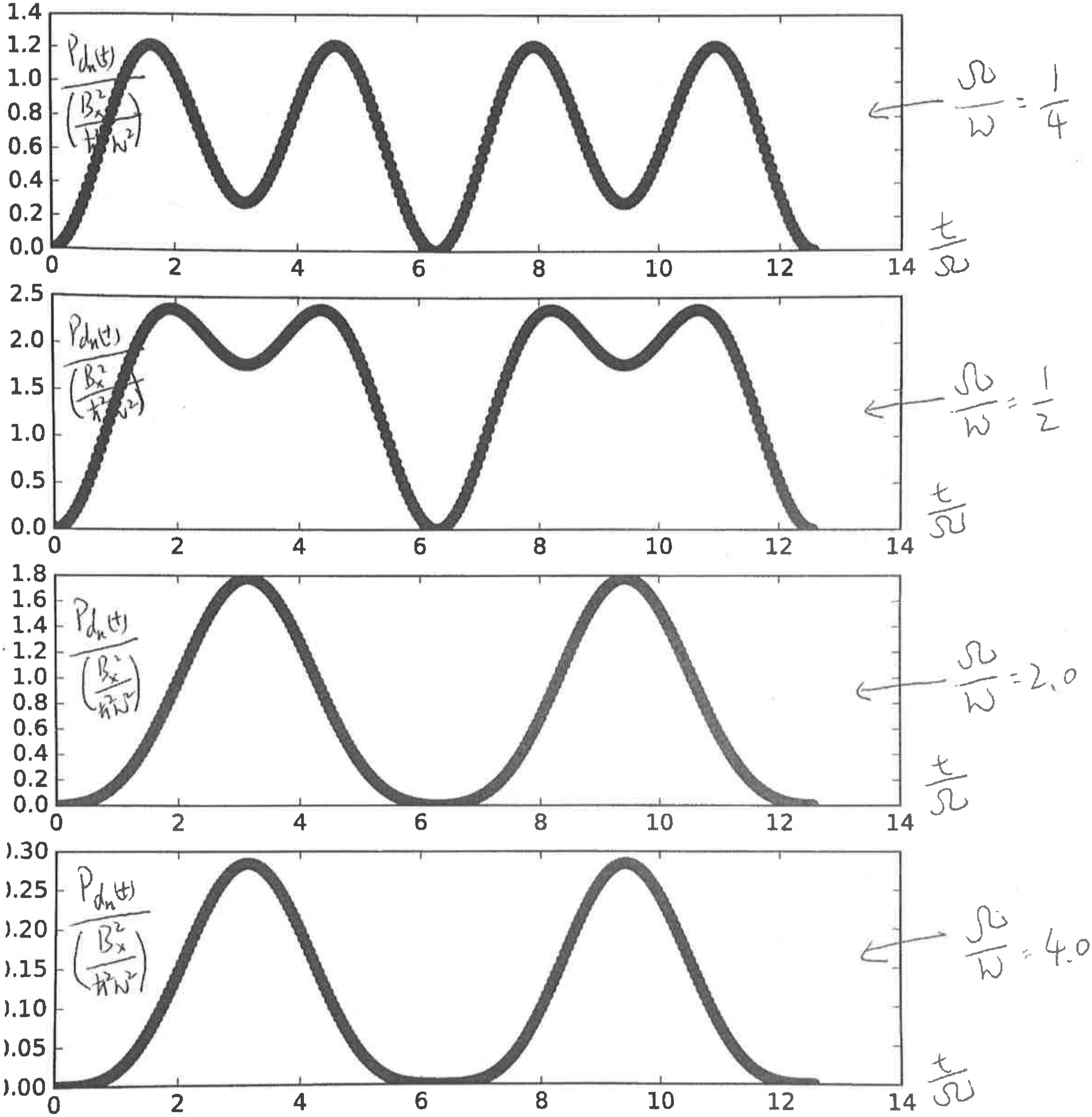
$$\vec{S}(t) = \langle \psi(t)|\vec{\sigma}|\psi(t)\rangle = \left[1, d^*_{dn}(t)\right](\sigma_x, \sigma_y, \sigma_z)\begin{bmatrix}1\\ d_{dn}(t)\end{bmatrix}$$

Denote $\begin{bmatrix}1\\ d_{dn}\end{bmatrix} = \begin{bmatrix}1\\ \epsilon\end{bmatrix}$ ← small

$$\left[1, \epsilon^*\right]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\ \epsilon\end{bmatrix} = \left[1, \epsilon^*\right]\begin{bmatrix}\epsilon\\ 1\end{bmatrix} = 2\,\mathrm{Re}\,\epsilon, \quad \left[1, \epsilon^*\right]\sigma_y\begin{bmatrix}1\\ \epsilon\end{bmatrix} = 0$$

So the vector direction of spin is $(2\,\mathrm{Re}\, d_{dn}(t), 0, 1)$.

The spin wobbles along $\hat{x}$ (away from $[0,0,1]$) according to $2\,\mathrm{Re}\, d_{dn}(t)$.



$$P_{d_n}(t) = \left| d_{d_n}(t) \right|^2 = \frac{B_x^2}{\hbar^2 \omega^2 \left[1 - \left(\frac{\Omega}{\omega}\right)^2\right]^2} \left[\sin^2 \Omega t + \left(\frac{\Omega}{\omega}\right)^2 (\cos \Omega t - 1)^2 \right]$$

$$\omega = \frac{\varepsilon_{up} - \varepsilon_{dn}}{\hbar} = \frac{B_z - (-B_z)}{\hbar} = \frac{2B_z}{\hbar}$$