PHY 215C. Homework #1.

Problem 1.

$$H' = -q\int_C \vec{E}\cdot d\vec{l} = -q\,\vec{r}\cdot\hat{k}\,\mathcal{E}\frac{\tau^2}{t^2+\tau^2} = -q\,r\cos\theta\,\mathcal{E}\frac{\tau^2}{t^2+\tau^2}$$

The ground state is $|100\rangle$ at $t=-\infty$, The final state is $|nlm\rangle$. From (18.2.9), we have

$$d_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_{-\infty}^{+\infty}\langle nlm|H'|100\rangle e^{i\omega_{fi}t'}dt'$$

$$= 0 - \frac{i}{\hbar}\langle nlm|r\cos\theta|100\rangle(-q\mathcal{E})\int_{-\infty}^{+\infty}\frac{\tau^2 e^{i\omega_{fi}t'}}{t'^2+\tau^2}dt'$$

$$= \frac{iq\mathcal{E}}{\hbar}\langle nlm|r\cos\theta|100\rangle\int_{-\infty}^{+\infty}\frac{\tau^2 e^{i\omega_{fi}t'}}{t'^2+\tau^2}dt'$$

Also, we know $n=2$ and for $n=2$, only $\langle 210|r\cos\theta|100\rangle$ is non zere, thus only $d_{|210\rangle}(t)$ is non zero.

Since $\langle 210|r\cos\theta|100\rangle = \sqrt{\frac{2^{15}}{3^{10}}}\,a_0$, we have

$$d_{|210\rangle}(t=+\infty) = \frac{iq\mathcal{E}}{\hbar}\left(\sqrt{\frac{2^{15}}{3^{10}}}\,a_0\right)\int_{-\infty}^{+\infty}\frac{\tau^2 e^{i\omega_{fi}t'}}{t'^2+\tau^2}dt'$$

$$= \frac{iq\mathcal{E}}{\hbar}\left(\sqrt{\frac{2^{15}}{3^{10}}}\,a_0\right)\int_{-\infty}^{+\infty}\frac{\tau^2 e^{i\omega_{fi}t'}}{t'^2+\tau^2}dt'$$

$$= \frac{iq\mathcal{E}}{\hbar}\sqrt{\frac{2^{15}}{3^{10}}}\,a_0\cdot 2\pi i\cdot(\text{Residue at } t'=i\tau)$$

$$= \frac{iq\mathcal{E}}{\hbar}\sqrt{\frac{2^{15}}{3^{10}}}\,a_0\,\pi\tau e^{-\omega_{fi}\tau}$$

$$\boxed{\omega_{fi} = \frac{E_f^0-E_i^0}{\hbar} = \frac{3m_e e^4}{8(4\pi\varepsilon_0)^2\hbar^3}}$$

(replacing q with $-e$)

$$= \frac{-ie\mathcal{E}}{\hbar}\sqrt{\frac{2^{15}}{3^{10}}}\,a_0\,\pi\tau\, e^{-\omega_{fi}\tau}$$

Thus, the probability ends up in $|210\rangle$ is $|d_{|210\rangle}(t=+\infty)|^2$ and the probability ends up in other states a is 0.

Our result: $P_{1\to 2} = \left(\frac{e\varepsilon}{\hbar}\right)^2 \left(\frac{2^{15} a_0^2}{3^{10}}\right) \pi^2 \tau^2 e^{-2W_{fi}\tau} \propto e^{-2W_{fi}\tau}$

Shankar's Exercise 18.2.2:

$$P_{1\to 2}^{\text{Shankar}} = \left(\frac{e\varepsilon}{\hbar}\right)^2 \left(\frac{2^{15} a_0^2}{3^{10}}\right) \pi \tau^2 e^{-W^2\tau^2/2} \propto e^{-W^2\tau^2/2}$$

Thus, there are essential differences. The decay factor is different. When τ gets large, $P_{1\to 2}^{\text{Shankar}}$ decays much faster.

<u>DISCUSSION</u>

- why is the fall-off exponential: $\tau e^{-\omega_{fi}\tau}$? when the integral is over an oscillating phase factor $e^{i\omega_{fi}t}$ and a Lorentzian $\frac{\tau^2}{t^2+\tau^2}$? Must be some unusual cancellation as ω_{fi}, or τ, get large.
- both expressions vanish as $\tau \to$ (with differing powers of τ) because the integrated "area" of $F(T)$ goes to zero.
- for a given ω_{fi}, there is a characteristic $\tau = \frac{1}{\omega_{fi}}$ for which $P_{1\to 2}$ has decreased by $1/e$. Higher excitation $\Rightarrow$ shorter τ is required.
- there is no reason why ω_{fi} can't be <u>negative</u>! Then the probability of <u>de-excitation</u> diverges as τ gets large! Is this correct? Anyway, 1st order surely breaks down.

Preblem 2.

$H' = -e\mathcal{E} X e^{-\frac{t^2}{\tau^2}}$ is applied between $t = -\tau$ and $t = +\tau$

thus we have

$$d_n(t=+\infty) = \frac{-i}{\hbar}\int_{-\tau}^{\tau} (-e\mathcal{E})\langle n|x|0\rangle e^{-\frac{t^2}{\tau^2}} e^{in\omega t}\, dt$$

Since $x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a+a^\dagger)$, only $d_1(t=+\infty)$ is non zero.

$$d_1(t=+\infty) = \frac{ie\mathcal{E}}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{1/2}\int_{-\tau}^{\tau} e^{-t^2/\tau^2} e^{i\omega t}\, dt$$

$$= \frac{ie\mathcal{E}}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{1/2} \cdot \frac{\sqrt{\pi}\,\tau e^{-\tau^2\omega^2/4}}{2}\left[\mathrm{erf}\left(1-\frac{i\tau\omega}{2}\right) + \mathrm{erf}\left(1+\frac{i\tau\omega}{2}\right)\right]$$

(erf stands for the "error function")

Thus, the probability is

$$P_{0\to1} = |d_1(t=\infty)|^2 = \frac{e^2\mathcal{E}^2\pi\tau^2}{8m\omega\hbar} e^{-\frac{\tau^2\omega^2}{2}}\left|\mathrm{erf}\left(1-\frac{i\tau\omega}{2}\right)+\mathrm{erf}\left(1+\frac{i\tau\omega}{2}\right)\right|^2$$

Shankar's Exercise: ~~18.2.4~~

$$P_{0\to1} = \frac{e^2\mathcal{E}^2\pi\tau^2}{2m\omega\hbar} e^{-\frac{\tau^2\omega^2}{2}}$$

<u>Discussion</u>. This result, with erf(z) w/ complex argument z, $z = 1 \pm i\frac{\omega\tau}{2}$, is pretty useless without a plot, or some knowledge of, and discussion of, its behavior vs. $\omega\tau$. A quick google (such as mathworld.wolfram.com/Erf.html) shows that is has very interesting behavior, but perhaps not in the region of our interest.

WEP.

Pr. 3. Projection Operators for $J_1=1$, $J_2=2$.

First, we need to know ↙ operating in total-J representation

$$\underset{\sim}{J_1}\cdot\underset{\sim}{J_2} = \tfrac{1}{2}\left[\underset{\sim}{J}^2-\underset{\sim}{J_1}^2-J_2^2\right] \rightarrow \tfrac{1}{2}\left[J(J+1) - J_1(J_1+1) - J_2(J_2+1)\right]$$

$$= \tfrac{1}{2}J(J+1) - \underbrace{\tfrac{1}{2}\cdot1\cdot2}_{-1} - \underbrace{\tfrac{1}{2}\cdot2\cdot3}_{-3} = \tfrac{1}{2}J(J+1)-4 \equiv K-4$$

K is just shorthand for $\frac{1}{2}J(J+1)$ which is always an integer

$$K_{J=3}=6,\quad K_{J=2}=3,\quad K_{J=1}=1.$$

Expand projector for each J, P_J, as

$$P_J = a_{J,1} + a_{J,2}\,\underset{\sim}{J_1}\cdot\underset{\sim}{J_2} + a_{J,3}\left(\underset{\sim}{J_1}\cdot\underset{\sim}{J_2}\right)^2$$

Write it out explicitly: $\underset{\sim}{J_1}\cdot\underset{\sim}{J_2} = K-4 = \begin{array}{c|c|c} J=1 & J=2 & J=3 \\ -3, & -1, & 2 \end{array}$

so

$$\begin{aligned} P_1(J=1) &= a_{11} - 3a_{12} + 9a_{13} = 1 \\ P_1(J=2) &= a_{11} - a_{12} + a_{13} = 0 \\ P_1(J=3) &= a_{11} + 2a_{12} + 4a_{13} = 0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & -3 & 9 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13}\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

so $\underset{\approx}{A}\,\underset{\sim}{a_1} = \underset{\sim}{X} \equiv \begin{bmatrix}1\\0\\0\end{bmatrix}$ so $\underset{\sim}{a_1} = \underset{\approx}{A}^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix}$

For P_2 $\quad \underset{\sim}{a_2} = \underset{\approx}{A}^{-1}\begin{bmatrix}0\\1\\0\end{bmatrix}$

For P_3 $\quad \underset{\sim}{a_3} = \underset{\approx}{A}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$

} repeated on next page.

so $\underset{\approx}{a} = \underset{\approx}{A}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \underset{\approx}{A}^{-1}$

so we have the matrix eq'n $\underset{\approx}{A}\,\underset{\sim}{a}_1 = \underset{\sim}{X} \equiv \begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \underset{\sim}{a}_1 = \underset{\approx}{A}^{-1}\begin{bmatrix}1\\0\\0\end{bmatrix}$.

For P_2 onto $J=2$ subspace
$$\underset{\approx}{A}\,\underset{\sim}{a}_2 = \underset{\sim}{Y} \equiv \begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \underset{\sim}{a}_2 = \underset{\approx}{A}^{-1}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

& for P_3
$$\underset{\approx}{A}\,\underset{\sim}{a}_3 = \underset{\sim}{Z} \equiv \begin{bmatrix}0\\0\\1\end{bmatrix} \quad \underset{\sim}{a}_3 = \underset{\approx}{A}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\underset{\approx}{A} = \begin{bmatrix}1 & -3 & 9\\ 1 & -1 & 1\\ 1 & 2 & 4\end{bmatrix}, \quad \underset{\approx}{A}^{-1} = \frac{1}{30}\begin{bmatrix}-6 & 30 & 6\\ -3 & -5 & 8\\ 3 & -5 & 2\end{bmatrix}$$

column vectors

$$\text{so } \left[\underset{\sim}{a}_1, \underset{\sim}{a}_2, \underset{\sim}{a}_3\right] = \underset{\approx}{A}^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \underset{\approx}{A}^{-1} \quad !!$$

$$\text{i.e. } \begin{Bmatrix}a_{1,1}\\ a_{1,2}\\ a_{1,3}\end{Bmatrix} = \begin{Bmatrix}-1/5\\ -1/10\\ 1/10\end{Bmatrix}, \quad \begin{Bmatrix}a_{2,1}\\ a_{2,2}\\ a_{2,3}\end{Bmatrix} = \begin{Bmatrix}1\\ -1/6\\ -1/6\end{Bmatrix}, \quad \begin{Bmatrix}a_{3,1}\\ a_{3,2}\\ a_{3,3}\end{Bmatrix} = \begin{Bmatrix}1/5\\ 4/15\\ 1/15\end{Bmatrix}$$

I.e.
$$P_{J=1} = -\frac{1}{5} - \frac{1}{10}\,\underset{\sim}{J}_1\cdot\underset{\sim}{J}_2 + \frac{1}{10}\left(\underset{\sim}{J}_1\cdot\underset{\sim}{J}_2\right)^2$$

$$\left.\begin{aligned}
&\xrightarrow{J=1} -\tfrac{1}{5} - \tfrac{1}{10}(-3) + \tfrac{1}{10}(9) = \frac{-2+3+9}{10} = 1 \ \checkmark\\
&\xrightarrow{J=2} -\tfrac{1}{5} - \tfrac{1}{10}(-1) + \tfrac{1}{10}(1) = \frac{-2+1+1}{10} = 0 \ \checkmark\\
&\xrightarrow{J=3} -\tfrac{1}{5} - \tfrac{1}{10}(2) + \tfrac{1}{10}(4) = \frac{-2-2+4}{10} = 0 \ \checkmark
\end{aligned}\right\} J=1 \text{ projector}$$

Notice: in P_J, J denotes the subspace of the projector on right hand side, J denotes subspace being operated on.

SPECIFICALLY:
$$P_{J=2} = 1 - \frac{1}{6}\,\underset{\sim}{J}_1\cdot\underset{\sim}{J}_2 - \frac{1}{6}\left(\underset{\sim}{J}_1\cdot\underset{\sim}{J}_2\right)^2.$$
$$P_{J=3} = 1/5 + \frac{4}{15}\,\underset{\sim}{J}_1\cdot\underset{\sim}{J}_2 + \frac{1}{15}\left(\underset{\sim}{J}_1\cdot\underset{\sim}{J}_2\right)^2.$$
Try them!