Lecture 2 : Berry Phase and Chern number

Berry Phase review

Assuming a physical system is depended on some parameters $\mathbf{R} = (R_1, R_2, \cdots, R_N)$, we have the *snapshot* Hamiltonian $H(\mathbf{R})$, its eigen-values and eigen-states:

$$H(\mathbf{R})|n(\mathbf{R})\rangle = E_n(\mathbf{R})|n(\mathbf{R})\rangle$$

where $|n(\mathbf{R})\rangle$ can have an arbitrary phase prefactor.

The parameters $\mathbf{R(t)}$ are slowly changed with time t, then the adiabatic evolution of time-dependent Schrodinger equation:

$$i\frac{d}{dt}|\psi(t)\rangle = H(\mathbf{R(t)})|\psi(t)\rangle$$

Take the Ansatz: $|\psi(t)\rangle = e^{i\gamma_n(t)}e^{-i\int_0^t E_n(\mathbf{R(t')})dt'}|n(\mathbf{R(t)})\rangle$, we have

$$-\left(\frac{d}{dt}\gamma_n\right)|n\rangle + i\left|\frac{d}{dt}n\right\rangle = 0$$

That gives the Berry phase expression:

$$\gamma_n(\mathcal{C}) = \int_{\mathcal{C}} i\langle n(\mathbf{R})|\nabla_{\mathbf{R}} n(\mathbf{R})\rangle d\mathbf{R}$$

Define *Berry connection*:

$$\mathbf{A}^{(n)}(\mathbf{R}) = i\langle n(\mathbf{R})|\nabla_{\mathbf{R}} n(\mathbf{R})\rangle = -Im\langle n(\mathbf{R})|\nabla_{\mathbf{R}} n(\mathbf{R})\rangle$$

Gauge transformation:

$$|n(\mathbf{R})\rangle \rightarrow e^{i\alpha(\mathbf{R})}|n(\mathbf{R})\rangle$$

$$\mathbf{A}^{(n)}(\mathbf{R}) \rightarrow \mathbf{A}^{(n)}(\mathbf{R}) - \nabla_{\mathbf{R}}\alpha(\mathbf{R})$$

$\gamma = \oint \mathbf{A}(\mathbf{R})d\mathbf{R}$ is gauge invariant.

Gauge and Parallel transportation: recalling the arbitrary phase

$$|n(\mathbf{R})\rangle \rightarrow e^{i\alpha(\mathbf{R})}|n(\mathbf{R})\rangle$$

why shouldn't we choose one which makes

$$\frac{d}{dt}|n\rangle \equiv 0$$

from

$$-\left(\frac{d}{dt}\gamma_n\right)|n\rangle + i\left|\frac{d}{dt}n\right\rangle = 0$$

then we have

$$\gamma_n = 0$$

There is no Berry Phase in this frame, which is called *inertial frame*, the condition $\frac{d}{dt}|n\rangle \equiv 0$ is called *parallel transportation*. All the information resorted to $|n(\mathbf{R})\rangle$, similar to a transformation from *active frame* to *passive frame*.

Berry curvature

Define the *Berry curvature*:

$$\mathbf{B}(\mathbf{R}) = \nabla_{\mathbf{R}} \times \mathbf{A}^{(n)}(\mathbf{R})$$

Using Stokes theorem, we have for the Berry Phase:

$$\gamma_n(\mathcal{C}) = \int_{\mathcal{S}} \mathbf{B}^{(n)}(\mathbf{R}) d\mathcal{S}$$

where $\mathcal{S}$ is any surface whose boundary is the loop $\mathcal{C}$.

Two useful formula:

- $B_j = \epsilon_{jkl}\partial_k A_l = -Im\epsilon_{jkl}\partial_k\langle n|\partial_l n\rangle = -Im\epsilon_{jkl}\langle\partial_k n|\partial_l n\rangle$, that is $\mathbf{B}^{(n)} = -Im\sum_{n'\neq n}\langle\nabla n|n'\rangle \times \langle n'|\nabla n\rangle$.

- $\mathbf{B}^{(n)} = -Im\sum_{n'\neq n}\langle\nabla n|n'\rangle \times \langle n'|\nabla n\rangle$, to calculate $\langle n'|\nabla n\rangle$, start from:

$$H(\mathbf{R})|n\rangle = E_n|n\rangle$$

$$\Rightarrow (\nabla H)|n\rangle + H|\nabla n\rangle = (\nabla E_n)|n\rangle + E_n|\nabla n\rangle$$

$$\Rightarrow \langle n'|\nabla H|n\rangle + \langle n'|H|\nabla n\rangle = E_n\langle n'|\nabla n\rangle$$

$$\Rightarrow \langle n'|\nabla n\rangle = \frac{\langle n'|\nabla H|n\rangle}{E_n - E_{n'}}$$

then we get:

$$\mathbf{B}^{(n)} = -Im\sum_{n'\neq n}\langle\nabla n|n'\rangle \times \langle n'|\nabla n\rangle = -Im\sum_{n'\neq n}\frac{\langle n|\nabla H|n'\rangle \times \langle n'|\nabla H|n\rangle}{(E_n - E_{n'})^2}$$

which is gauge invariant!

Berry curvature from perturbation theory

We can use time-independent perturbation theory to derive the changes of instant snapshot basis:

$$H(\mathbf{R})|n(\mathbf{R})\rangle = E_n(\mathbf{R})|n(\mathbf{R})\rangle$$

we have

$$|n(\mathbf{R}+\mathbf{\Delta R})\rangle = |n(\mathbf{R})\rangle + \sum_{m\neq n} \frac{\langle m|H(\mathbf{R}+\mathbf{\Delta R}) - H(\mathbf{R})|n\rangle}{E_n - E_m}|m(\mathbf{R})\rangle$$

We see that $\langle n|\Delta n(\mathbf{R})\rangle = 0$, which means we have used *parallel transport* gauge, more general, we should add a arbitrary phase factor in the above equation for $|n(\mathbf{R}+\mathbf{\Delta R})\rangle$.

$$\nabla_{\mathbf{R}}|n\rangle = \sum_{m\neq n} \frac{\langle m|\nabla_{\mathbf{R}} H|n\rangle}{E_n - E_m}|m\rangle$$

From $\mathbf{B}^{(n)} = -Im \sum_{n'\neq n} \langle \nabla n|n'\rangle \times \langle n'|\nabla n\rangle$, we also get:

$$\mathbf{B}^{(n)} = -Im \sum_{n'\neq n} \frac{\langle n|\nabla H|n'\rangle \times \langle n'|\nabla H|n\rangle}{(E_n - E_{n'})^2}$$

Also notice:

$$\begin{aligned}\sum_n \mathbf{B}^{(n)} &= - Im \sum_n \sum_{n'\neq n} \frac{\langle n|\nabla H|n'\rangle \times \langle n'|\nabla H|n\rangle}{(E_n - E_{n'})^2} \\ &= - Im \sum_n \sum_{n'<n} \frac{\langle n|\nabla H|n'\rangle \times \langle n'|\nabla H|n\rangle + \langle n'|\nabla H|n\rangle \times \langle n|\nabla H|n'\rangle}{(E_n - E_{n'})^2} \\ &= 0\end{aligned}$$

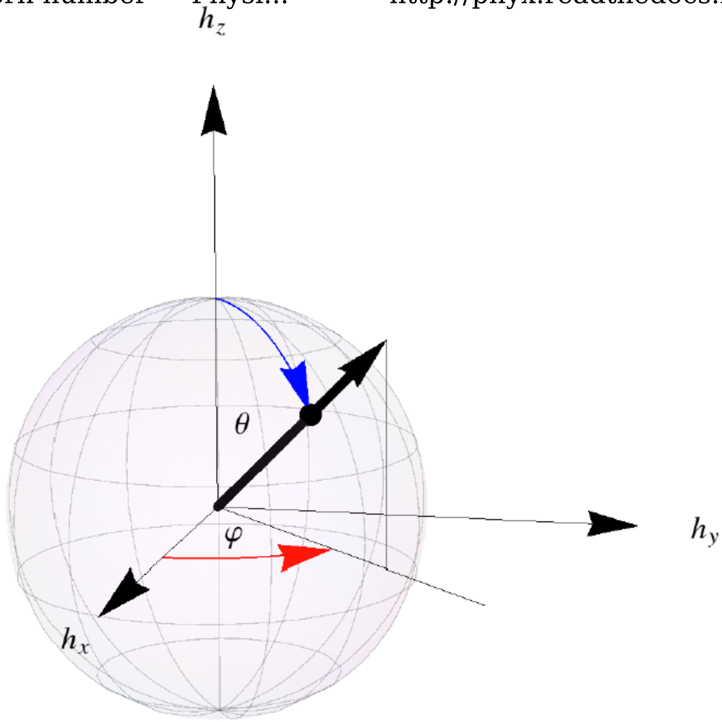
Which gives:

$$\sum_n \gamma_n(\mathcal{C}) = \int_{\mathcal{S}} \sum_n \mathbf{B}^{(n)}(\mathbf{R}) d\mathcal{S} = 0$$

Benchmark: Spin-1/2

Gauge!Gauge!Gauge!

2-level Hamiltonian $H(\mathbf{R}) = h_0(\mathbf{R})\sigma_0 + \mathbf{h}(\mathbf{R})\cdot\sigma$, we can set $h_0 = 0$, because it does not affect the eigenstates, eigen-energy are $\pm|\mathbf{h}|$, introduce the unit vector:$\hat{\mathbf{h}} = \mathbf{h}/|\mathbf{h}|$, the endpoints of $\hat{\mathbf{h}}$ map out the surface of a unit sphere, called the *Bloch sphere* shows below:



Bloch sphere