Physics 140A: TakeHome Exam


1. Vibrating Chain of Atoms. 15 points.
Consider a chain of identical atoms of mass M, spacing a.
(a) Write the form of solution for the atomic displacements that we use to obtain wavelike solutions to Newton’s 2nd law.
(b) Draw a diagram and label atoms. Then consider a chain whose 2nd neighbor force constant it $K_2$, and 3rd neighbor force constant is $K_3$ (no nearest neighbor spring). Determine the phonon dispersion relation $\omega(k)$ and calculate the sound velocity ($k \to 0$).
(c) Is the chain stable for every positive value of $K_2$ and $K_3$?

2. Debye Model of Phonons. 20 points.
Use the Debye phonon dispersion $\omega(k) = \frac{c|k|}{k_D}$ for a 3D simple cubic lattice.
(a) Determine the Debye wavevector $k_D$ such that the “Debye sphere” contains the same number of states as the Brillouin zone does. Express it in terms of the lattice constant a.
(b) Evaluate the density of states per primitive cell $\rho(\omega)$. Assume that all three branches are “degenerate,” that is, they all have the same energy, i.e. they are identical.
(c) Sketch the result, showing the correct behavior both at $\omega \to 0$ and $\omega \to \omega_D \equiv ck_D$.

3. Einstein Oscillators and Internal Energy. 20 points.
(a) Write the Bose-Einstein thermal occupation function, defining what it is and what enters it.
(b) Consider an Einstein optical branch $\omega(k) = \Omega$, a constant. First write the density of states for a system of $N$ oscillators in a volume V.
(c) Then evaluate the energy $U(T)$ of the vibrational system. Point out its high temperature and low temperature limits, and argue why these results make physical sense.
4. **Debye Specific Heat.** 5 points.  
Given: in 2D the phonon density of states of an elemental simple square lattice is $\rho(\omega) = P\omega$ per unit cell, in the Debye approximation.  
(a) Determine the constant $P$ such that the “Debye circle” contains the correct number of states.  
(b) Write the expression in general form for the heat capacity per unit cell, in terms of the dispersion relation, cell parameters, etc.  
(c) What is the $T$-dependence of the phonon heat capacity for a 3D solid in the low-$T$ limit (which is given correctly by the Debye model). This is something that you should remember, and all that is wanted is the answer; you probably won’t have time to work it out and are not expected to.

5. **Free Electron Gas (FEG).** 10 points.  
Consider a 1D free electron gas.  
(a) Consider the energy derivative $df/d\varepsilon$ of the Fermi-Dirac thermal distribution function in the $T=0$ case. Show that it is proportional to the $\delta$-function $\delta(\varepsilon - \varepsilon_F)$, and find the proportionality constant. Refer to the defining properties of the $\delta$-function.  
(b) Write the FEG dispersion relation, then calculate the density of states.  
(c) Suppose the electron density of states for some system has the form $\rho(E) = AE^g$, where $g$ is a positive constant. For a Fermi energy $E_F >> k_BT$, what is the $T$-dependence of the heat capacity to lowest order (smallest exponent of $T$)? Hint: this is a rather simple question, don’t do a lot of calculating because you don’t need to.

6. **Liquid Helium.** 10 points.  
Liquid $^3$He (helium) is a fermion (spin 1/2) and, perhaps surprisingly, is a very good “free fermion gas”, i.e. free electron gas except no electric charge. But we have neglected interactions anyway. Its mass density is 0.08 g/cm$^3$. Calculate the Fermi wavevector $k_F$ and Fermi energy $E_F$; what is the corresponding Fermi temperature? Helpful numbers: $\hbar \approx 10^{-27}$ erg s, electron mass $m \approx 0.9 \times 10^{-27}$ g. Also, $\hbar/m \approx 1.2 \ cm^2/s$, and 1 eV = $1.6\times10^{-12}$ erg. Make numerical approximations, no calculator, answer within a factor of ten is fine. *Remember, this is $^3$He!*