PHYS 140A HW#5

1. (a) $\omega(k) = ck - \frac{1}{2}\frac{ca}{\pi}k^2$, $\quad \underset{\sim}{v}(k) = \nabla_k\, \omega(k) = (c - \frac{ca}{\pi}k)\,\hat{k}$

(b) $\omega = ck - \frac{ca}{2\pi}k^2$ or $\omega a = cka - \frac{c}{2\pi}(ka)^2$: $\frac{1}{2\pi}\xi^2 - \xi + \frac{\omega a}{c} = 0$

Let $\omega a/c \equiv \xi_0$

$$\xi = \left(1 \pm \sqrt{1 - \frac{4\xi_0}{2\pi}}\right)\pi = \pi\left(1 - \sqrt{1 - 2\xi_0/\pi}\right)$$

since $\xi = ka$ should $\to 0$ as $\xi_0(\sim\omega) \to 0$

Thus $\frac{1}{\pi}\xi = k\frac{a}{\pi} = k/(\pi/a) = 1 - \sqrt{1 - \frac{2a}{c\pi}\omega} \equiv 1 - \sqrt{1 - \omega/\omega_0} = k(\omega)\, a/\pi$

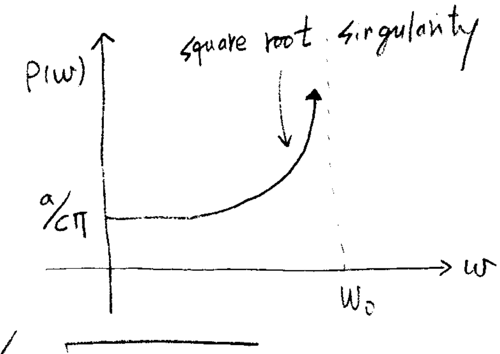
(c) 1D

$$\rho(\omega) = \sum_k \delta(\omega - \omega(k)) = \frac{a}{2\pi}\int \frac{\delta(k - k(\omega))\, dk}{|c - \frac{ca}{\pi}k(\omega)|}$$ satisfied at $\pm k(\omega)$

$$= \frac{a}{\pi}\left\{c - \left[c - \frac{cak}{\pi}\right]\right\}^{-1} = \frac{a}{\pi}\frac{\pi}{cak(\omega)} = \frac{a}{c\pi}\frac{1}{\sqrt{1 - \omega/\omega_0}}$$

What is $\omega_0 \equiv \frac{\pi c}{2a}$? $\quad \omega(\frac{\pi}{a}) = c\frac{\pi}{a} - \frac{1}{2}c\frac{a}{\pi}\frac{\pi}{a}\frac{\pi}{a} = \frac{c\pi}{2a} \equiv \omega_0$

$$v_k \to v(\omega) = c\left(1 - \frac{ak(\omega)}{\pi}\right) = c\sqrt{1 - \omega/\omega_0}$$



(d) 2D

$$\rho(\omega) = \frac{a^2}{4\pi^2}\int_{\text{circle}(\omega)} \frac{k(\omega)}{|v(\omega)|}$$

$$= \frac{a^2}{2\pi}\frac{k(\omega)}{v(\omega)} = \frac{a}{2}\left[1 - \sqrt{1 - \omega/\omega_0}\right] / c\sqrt{1 - \omega/\omega_0}$$

$$\rho(\omega) = \frac{a}{2c}\left\{\frac{1}{\sqrt{1 - \omega/\omega_0}} - 1\right\}$$

As $\omega \to 0$, $\quad (1 - \omega/\omega_0)^{-\frac{1}{2}} \simeq 1 + \omega/2\omega_0 + \cdots$

So $\rho(\omega) \longrightarrow \frac{a}{2c}\cdot\frac{\omega}{2\omega_0} = \frac{a}{4c\omega_0}\omega$ as $\omega \to 0$

As $\omega \to \omega_0$, again $\rho(\omega)$ has an inverse square root singularity.

2. For $a>0$

$$\int \delta(x) f(x)\, dx = f(0) \quad —①$$

$$\int \delta(ax) f(x)\, dx = \int \delta(y) f\left(\frac{y}{a}\right) \frac{dy}{a} = \frac{1}{a} f(0) \qquad \text{Let } y = ax,\ x = \frac{y}{a}$$

$$\rightarrow \int a\, \delta(ax)\, f(x)\, dx = f(0)$$

compare with ①, we have $\delta(ax) = \frac{\delta(x)}{a}$ $(a>0)$

For $a<0$, let $a=-|a|$

$$\int \delta(-|a|x) f(x)\, dx = \int \delta(-y) f\left(\frac{y}{|a|}\right) \frac{dy}{|a|} = \frac{1}{|a|} f(0) \qquad \text{Let } y = |a|x,\ x = \frac{y}{|a|}$$

$$\rightarrow \int |a|\, \delta(ax) f(x)\, dx = f(0)$$

compare with ①, we have $\delta(ax) = \frac{\delta(x)}{|a|}$ $(a<0)$

$$\therefore\ \delta(ax) = \frac{\delta(x)}{|a|}$$

3. $E = N \langle n \rangle \hbar \omega$

$$= N \cdot \frac{1}{e^{\hbar\omega/k_BT} - 1} \hbar\omega$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = N\hbar\omega \cdot \frac{-e^{\hbar\omega/k_BT} \cdot \frac{\hbar\omega}{k_B T} \cdot (-1) \cdot T^{-2}}{(e^{\hbar\omega/k_BT} - 1)^2}$$

$$= \frac{N k_B \left(\frac{\hbar\omega}{k_BT}\right)^2 e^{\hbar\omega/k_BT}}{(e^{\hbar\omega/k_BT} - 1)^2}$$

At high temperature, $C_V = N k_B\, x^2 (1 + x + x^2/2! + \cdots) \Big/ (x + x^2/2! + \cdots)^2$

$\left(x \equiv \frac{\hbar\omega}{k_BT} \rightarrow 0\right)$

$$\longrightarrow N k_B$$

At low temperature, $C_V = \frac{N k_B\, x^2 e^x}{(e^x - 1)^2} \propto e^{-x} = e^{-\hbar\omega/k_BT}$

$$\Rightarrow \begin{cases} T \rightarrow \infty, & C_V \rightarrow N k_B \\ T \rightarrow 0, & C_V \propto e^{-\hbar\omega/k_BT} \end{cases}$$

4.

$$N = \left(\frac{L}{2\pi}\right)^2 \pi k^2 \longrightarrow k_D = \sqrt{\frac{4\pi N}{L^2}}, \quad \omega_D = k_D v = \frac{\sqrt{4\pi N}\, v}{L},$$

$$D(\omega) = \frac{dN}{d\omega} = \frac{dN}{dk}\frac{dk}{d\omega} = \left(\frac{L}{2\pi}\right)^2 \cdot 2\pi \frac{\omega}{v} \cdot \frac{1}{v}$$

$$= \left(\frac{L}{2\pi}\right)^2 \frac{2\pi \omega}{v^2}$$

Thermal energy :

$$U = 2\int d\omega\, D(\omega) \langle n(\omega)\rangle \hbar\omega$$

two branches

$$= 4\pi \left(\frac{L}{2\pi}\right)^2 \frac{\hbar}{v^2} \int_0^{\omega_D} \frac{\omega^2}{e^{\hbar\omega/k_BT} - 1}\, d\omega$$

Let $x = \hbar\omega/k_BT$, $\quad x_D \equiv \frac{\hbar\omega_D}{k_BT} = \frac{\Theta}{T}$

$$\Theta^2 = \frac{\hbar^2 \omega_D^2}{k_B^2} = \left(\frac{2\pi}{L}\right)^2 \frac{\hbar^2 v^2 N}{\pi k_B^2}$$

$$\Rightarrow \quad U = 4\pi \left(\frac{L}{2\pi}\right)^2 \frac{k_B^3 T^3}{\hbar^2 v^2} \int_0^{x_D} \frac{x^2}{e^x - 1}\, dx$$

$$= 4N k_B \frac{T^3}{\Theta^2} \int_0^{x_D} \frac{x^2}{e^x - 1}\, dx$$

As $T \ll \Theta$ (very low temperature), $x_D \to \infty$

$$U = 4Nk_B \frac{T^3}{\Theta^2} \int_0^{\infty} \frac{x^2}{e^x - 1}\, dx$$

$\equiv$ number

$$\therefore \quad C_v = \frac{\partial U}{\partial T} = 12 N k_B \left(\frac{T}{\Theta}\right)^2 \int_0^{\infty} \frac{x^2}{e^x - 1}\, dx$$

$\sim 28.85\, N k_B \left(\frac{T}{\Theta}\right)^2$, depends on T^2 instead of T^3 in 3-D

* compare to 3-D, $C_v \sim 234 N k_B \left(\frac{T}{\Theta}\right)^3$