3. hcp structure

(a) Use $\vec{R}_1$, $\vec{R}_2$, $\vec{R}_3$ as the direct lattice vectors.

Reciprocal vectors
$$\vec{G}_1 = 2\pi \frac{\vec{R}_2 \times \vec{R}_3}{\vec{R}_1 \cdot (\vec{R}_2 \times \vec{R}_3)}$$
$$\vec{G}_2 = 2\pi \frac{\vec{R}_3 \times \vec{R}_1}{\vec{R}_1 \cdot (\vec{R}_2 \times \vec{R}_3)}$$
$$\vec{G}_3 = 2\pi \frac{\vec{R}_1 \times \vec{R}_2}{\vec{R}_1 \cdot (\vec{R}_2 \times \vec{R}_3)}$$

In HW2,
$$\begin{cases} \vec{R}_1 = a\left(\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}\right) \\ \vec{R}_2 = a\left(-\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}\right) \\ \vec{R}_3 = c\hat{z} \end{cases} \rightarrow \begin{cases} \vec{G}_1 = \frac{4\pi}{\sqrt{3}a}\left(\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) \\ \vec{G}_2 = \frac{4\pi}{\sqrt{3}a}\left(-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right) \\ \vec{G}_3 = \frac{2\pi}{c}\hat{z} \end{cases}$$

(b) $\vec{G} = h\vec{G}_1 + k\vec{G}_2 + l\vec{G}_3$, two atoms at $(0,0,0)$, $\left(\frac{2}{3}\vec{R}_1 + \frac{1}{3}\vec{R}_2 + \frac{1}{2}\vec{R}_3\right)$

$$\Rightarrow S_G = f\left[1 + \exp\left[-i\left(\frac{2}{3}\vec{R}_1 + \frac{1}{3}\vec{R}_2 + \frac{1}{2}\vec{R}_3\right)\cdot\left(h\vec{G}_1 + k\vec{G}_2 + l\vec{G}_3\right)\right]\right]$$

$$\vec{R}_1 \cdot \vec{G}_1 = 2\pi \frac{\vec{R}_1 \cdot (\vec{R}_2 \times \vec{R}_3)}{\vec{R}_1 \cdot (\vec{R}_2 \times \vec{R}_3)} = 2\pi$$

$$\vec{R}_2 \cdot \vec{G}_2 = \vec{R}_3 \cdot \vec{G}_3 = 2\pi$$

$$\vec{R}_1 \cdot \vec{G}_2 = \vec{R}_1 \cdot \vec{G}_3 = \vec{R}_2 \cdot \vec{G}_1 = \vec{R}_2 \cdot \vec{G}_3 = \vec{R}_3 \cdot \vec{G}_1 = \vec{R}_3 \cdot \vec{G}_2 = 0$$

$$\Rightarrow S_G = f\left[1 + \exp\left[-i\left(\frac{2}{3}h \cdot 2\pi + \frac{1}{3}k \cdot 2\pi + \frac{1}{2}l \cdot 2\pi\right)\right]\right]$$
$$= f\left[1 + e^{-i\pi\left(\frac{4}{3}h + \frac{2}{3}k + l\right)}\right]$$